Extremal Black Hole Entropy

Motivation:

Low energy limit of string theory gives rise to gravity coupled to other fields.

These theories typically have black hole solutions.

Thus string theory gives a framework for studying classical and quantum properties of black holes.

One of the important properties characterizing a black hole is the Bekenstein-Hawking entropy S_{BH}.

In the low energy limit

$$S_{BH} = A/(4G_N)$$

For a wide class of extremal black holes

$$S_{BH} = S_{stat}, \qquad S_{stat} \equiv \ln(Degeneracy)$$

Strominger, Vafa; . . .

This gives a good understanding of this entropy from microscopic viewpoint.

Originally the comparison between black hole and statistical entropy was carried out in the limit of large charges.

In this limit the curvature at the horizon is small and hence we can ignore higher derivative corrections to the effective action in computing the black hole entropy.

On the microscopic side we can use appropriate asymptotic formula for the degeneracy of states to calculate the statistical entropy.

Given this success, it is natural to carry out our study of black holes to finer details.

What are the effects of higher derivative corrections to the black hole entropy?

Does the agreement between black hole entropy and statistical entropy continue to hold even after taking into account the effect of these higher derivative corrections?

In order to attack this problem we need to open two fronts.

First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy.

But we also need to know how to calculate the statistical entropy to greater accuracy.

In this talk we shall address the first problem.

In the next talk we shall address the second problem.

References for this talk

A.S. hep-th/0506177, hep-th/0508042

Bindusar Sahoo and A.S., hep-th/0601228, hep-th/0603149

Astefanesei,Goldstein, Jena, A.S., Trivedi, hep-th/0606244

A general frameork for computing higher derivative corrections to black hole entropy has been developed by Wald.

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

for spherically symmetric black holes.

In computing $\delta\mathcal{S}/\delta R_{\mu\nu\rho\sigma}$

1. express the action $\mathcal{S}$ in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

Our goal:

Develop a general method for calculating higher derivative corrections to S_{BH} for extremal, but not necessarily supersymmetric black holes.

How do we define extremal black holes in a higher derivative theory?

Take the clue from usual (super-)gravity.

We shall consider spherically symmetric extremal black holes in $D = 4$ but the analysis can be easily generalized to

1. Spherically symmetric black holes in other dimensions

2. Rotating black holes and black rings

Reissner-Nordstrom solution in $D = 4$:

$$\begin{aligned} ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)dt^2 \\ &\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\ &\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Extremal limit: $\rho_+ = \rho_-$

Define $\tau = \lambda\, t/\rho_+^2, \qquad r = (\rho - \rho_+)/\lambda,$

$$\begin{aligned} ds^2 &= -\frac{r^2\rho_+^4}{(\rho_+ + \lambda r)^2}d\tau^2 + \frac{(\rho_+ + \lambda r)^2}{r^2}dr^2 \\ &\quad + (\rho_+ + \lambda r)^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

$$ds^2 = -\frac{r^2\rho_+^4}{(\rho_+ + \lambda r)^2}d\tau^2 + \frac{(\rho_+ + \lambda r)^2}{r^2}dr^2$$
$$+(\rho_+ + \lambda r)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Take the 'near horizon limit' $\lambda \to 0$.

$$ds^2 = \rho_+^2(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$\to$ near horizon geometry $AdS_2 \times S^2$

Isometry group: $SO(2,1) \times SO(3)$

The complete near horizon solution:

$$ds^2 = \rho_+^2(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$F_{rt} = \frac{q}{4\pi}, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta$$

$$\rho_+^2 = G_N \frac{q^2 + p^2}{4\pi}$$

q, p: label electric and magnetic charges

The full background has $SO(2,1)\times SO(3)$ isometry.

All known spherically symmetric extremal black holes in four dimensions with non-singular horizon have near horizon field configuration with $SO(2,1) \times SO(3)$ isometry.

(These include some solutions in the presence of certain higher derivative terms.)

We shall take this as the definition of extremal black holes.

In $D = 4$ we define an extremal non-rotating black hole to be one whose near horizon geometry and other field configurations have

$$SO(2,1) \times SO(3)$$

isometry.

Generalizations:

1. A rotating extremal black hole in D=4 has near horizon geometry with $SO(2,1) \times U(1)$ isometry

2. A non-rotating extremal black hole in general D has near horizon geometry with $SO(2,1) \times SO(D-1)$ isometry.

The entropy of an extremal black hole

$\equiv$ entropy of a non-extremal black hole in the extremal limit.

Thus we can use Wald's formula for the entropy for a non-extremal black hole with regular horizon.

Consider an arbitrary general coordinate invariant theory of gravity coupled to a set of Maxwell fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$.

The most general form of the near horizon geometry of an extremal black hole consistent with $SO(2,1) \times SO(3)$ isometry:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F_{rt}^{(i)} = e_i, \qquad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi} \sin\theta \,,$$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s \qquad F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

v_1, v_2: sizes of AdS_2 and S^2

u_s: scalar field values at the horizon.

$p_i/4\pi$: near horizon radial magnetic field

e_i: near horizon radial electric field

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right)$$

$$+ v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$\phi_s = u_s \qquad F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta\,,$$

$$R_{\alpha\beta\gamma\delta} = -v_1 (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \quad \alpha, \beta, \gamma, \delta = r, t$$

$$R_{mnpq} = v_2 (g_{mp} g_{nq} - g_{mq} g_{np}), \quad m, n, p, q = \theta, \phi$$

For this background covariant derivatives of the Riemann tensor, scalar fields and gauge field strengths vanish.

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \, ,$$

Let $\sqrt{-\det g}\,\mathcal{L}$ be the Lagrangian density.

Define:

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \sqrt{-\det g}\, \mathcal{L}$$

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\,\pi (e_i \, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p}))$$

Results:

For an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$,

1. the values of $\{u_s\}$, $\{e_i\}$, v_1 and v_2 are obtained by extremizing $\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})$ with respect to these variables.

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0$$

2. $S_{BH} = \mathcal{E}$ at the extremum.

1. The results are derived using the equations of motion and Wald's formula for entropy in the presence of higher derivative terms in the action.

The derivation does not require the theory and/or the solution to be supersymmetric.

2. The only requirements are gauge and general coordinate invariance of the Lagrangian density $\mathcal{L}$.

3. Similar results hold for rotating black holes and black holes in higher dimensions.

To summarize, the single ‘entropy function’ $\mathcal{E}$ determines

– the near horizon values $\{u_s\}$ of the scalar fields,

– the sizes v_1, v_2 of AdS_2 and S^2

– the gauge field strengths $\{e_i\}$

– the entropy S_{BH}

These results are useful for explicit calculations as well as proving general results.

The entropy function formalism leads to a proof of the ‘generalized attractor mechanism’ for extremal black holes.

The entropy of an extremal black hole depends only on its charges and is independent of all other asymptotic data *e.g.* the vev of the moduli scalar fields.

Proof:

If $\mathcal{E}$ has no flat directions then the extremization of $\mathcal{E}$ determines $\vec{u}$, $\vec{v}$, $\vec{e}$ completely in terms of $\vec{q}$, $\vec{p}$.

$\rightarrow S_{BH} = \mathcal{E}$ is independent of any other asymptotic data.

If $\mathcal{E}$ has flat directions, then extremization of $\mathcal{E}$ does not determine $\vec{u}$, $\vec{v}$, $\vec{e}$ uniquely.

But since $\mathcal{E}$ does not depend on the flat directions, $S_{BH} = \mathcal{E}$ is still determined in terms of $\vec{q}$, $\vec{p}$ and is independent of any other asymptotic data.

Entropy computation using entropy function

Take Einstein-Maxwell theory in $D = 4$:

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Consider an extremal black hole solution with near horizon geometry:

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$F_{rt} = e, \qquad F_{\theta\phi} = p \sin\theta / 4\pi$$

Then

$$
\begin{aligned}
f(v_1, v_2, e, p) &= \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L} \\
&= 4\pi\, v_1 v_2 \left[\frac{1}{16\pi G_N}\left(-\frac{2}{v_1} + \frac{2}{v_2}\right)\right. \\
&\qquad \left. + \frac{1}{2}\, v_1^{-2} e^2 - \frac{1}{2} v_2^{-2} \left(\frac{p}{4\pi}\right)^2\right] .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(v_1, v_2, e, q, p) &= 2\pi(q\, e - f) \\
&= 2\pi \left[q\, e - \frac{1}{4G_N}(2v_1 - 2v_2)\right. \\
&\left. -2\pi\ v_2\, v_1^{-1}\, e^2 + 2\pi\, v_1\, v_2^{-1} \left(\frac{p}{4\pi}\right)^2\right] .
\end{aligned}
$$

$$\mathcal{E}(v_1, v_2, e, q, p) = 2\pi \left[q\, e - \frac{1}{4G_N}(2v_1 - 2v_2) \right.$$
$$\left. -2\pi\ v_2\, v_1^{-1}\, e^2 + 2\pi\, v_1\, v_2^{-1} \left(\frac{p}{4\pi}\right)^2 \right] .$$

$\partial\mathcal{E}/\partial e = 0, \quad \partial\mathcal{E}/\partial v_1 = 0, \quad \partial\mathcal{E}/\partial v_2 = 0$ gives

$$q = 4\pi\, v_2 v_1^{-1} e, \qquad v_1 = v_2 = G_N\, \frac{q^2 + p^2}{4\pi} .$$

$$S_{BH} = \mathcal{E} = \frac{1}{4}(q^2 + p^2)$$

$\rightarrow$ correct answer for the entropy of extremal charged black holes.

Application to $\mathbb{Z}_N$ CHL models

1. Take heterotic string theory on T^6.

2. Take an orbifold by a $\mathbb{Z}_N$ group which preserves $\mathcal{N} = 4$ supersymmetry.

Equivalently we can also regard this as a $\mathbb{Z}_N$ orbifold of type IIA string theory on $K3 \times T^2$.

The low energy effective action of this theory is $\mathcal{N} = 4$ supergravity coupled to certain matter fields.

Higher derivative corrections include a Gauss-Bonnet term at the four derivative level:

$$\sqrt{-\det G}\Delta\mathcal{L}$$

$$= \phi(a, S)\sqrt{-\det g}\left\{R_{g\mu\nu\rho\sigma}R_g^{\mu\nu\rho\sigma} - 4R_{g\mu\nu}R_g^{\mu\nu} + R_g^2\right\}$$

a, S: axion-dilaton fields

For $N = 1, 2, 3, 5, 7$ we have

$$\phi(a, S) = -\frac{1}{64\pi^2}\Big((k+2)\ln S + \ln f^{(k)}(a+iS) + \ln f^{(k)}(-a+iS)\Big)$$

$$k = \frac{24}{N+1} - 2$$

$$f^{(k)}(\tau) = \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}$$

Given the action it is easy to compute the entropy function of a black hole with electric charge $\vec{Q}$ and magnetic charge $\vec{P}$.

After eliminating all the near horizon parameters except the near horizon values (u_a, u_S) of (a, S), we get

$$\mathcal{E} = \frac{\pi}{2}\Big[\left(\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q\cdot P\right) + 128\,\pi\,\phi(u_a, u_S)\Big]$$

u_a, u_S: near horizon values of a, S

Q^2, P^2, $Q \cdot P$: T-duality invariant inner products.

$\mathcal{E}$ has to be extremized with respect to u_a, u_S.

In the supergravity approximation $\phi(a, S) = 0$.

$$\mathcal{E} = \frac{\pi}{2}\left[\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q \cdot P\right]$$

is extremized at

$$u_S = \frac{\sqrt{Q^2P^2 - (Q \cdot P)^2}}{P^2}, \qquad u_a = \frac{Q \cdot P}{P^2}$$

$$S_{BH} = \mathcal{E}_{\text{extremum}} = \pi\sqrt{Q^2P^2 - (Q \cdot P)^2}$$

$\rightarrow$ reproduces known answer.

Special case

$Q \cdot P = 0$, $Q^2 >> P^2 >> 1$

In this case in the supergravity approximation $u_a = 0$, $u_S \simeq \sqrt{Q^2/P^2} >> 1$.

$\rightarrow$ weak heterotic string coupling.

$\rightarrow$ in studying the effect of higher derivative corrections we can use heterotic tree level result

Let us now return to the Gauss-Bonnet term.

Although we have found the expression for the entropy in the presence of the Gauss-Bonnet term, we have overlooked an important issue.

Even at the level of four derivatives the effective action contains many other terms besides the Gauss-Bonnet term.

What is their effect on the entropy?

Although we do not know the complete answer to this question, we know that at least in a special case the Gauss-Bonnet term gives the complete result for the entropy.

Consider the case $Q \cdot P = 0$, $Q^2 >> P^2 >> 1$

In this case we have weak heterotic string coupling and hence can use heterotic tree level result for studying higher derivative corrections.

At heterotic tree level

$$\phi(u_a, u_S) \simeq u_S/16\pi$$

Extremization of $\mathcal{E}$ gives

$$\rightarrow \quad S_{BH} \simeq \pi\sqrt{Q^2}\sqrt{P^2+8}$$

This is the result of including only the Gauss-Bonnet term.

In this case we can repeat the analysis by including the set of all tree level four derivative correction terms in the Lagrangian.

Sahoo,Sen;Exirifard

Result: Same as the one obtained by just using the Gauss-Bonnet term.

One can also give a general argument based on supersymmetry that tree level higher derivative terms do not modify the result. Kraus, Larsen

When Q and P are of same order, then keeping only tree level terms is not a useful approximation scheme.

Thus we need to include the full $\phi(a, S)$ as coefficient of the Gauss-Bonnet term.

But in order to be consistent we must also include other four derivative terms in the quantum effective action.

Does the effect of these additional terms on the entropy vanish like their classical counterpart?

As of now there is no known entropy non-renormalization theorem for loop corrections in the heterotic theory.

We shall proceed with the assumption that at least at the level of four derivative terms, the result for entropy obtained by including the Gauss-Bonnet term is exact.

Question: Can we find an exact formula for the degeneracy $d(Q,P)$ of these dyonic back holes using a microscopic description and compare the black hole entropy with $\ln d(Q,P)$?