Non-Metric Quantum Gravity

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Motivations

Develop a new perspective on (non)-renormalizability of perturbative quantum gravity. To this end, use certain “non-metric” (but classically equivalent) formulation.

This talk

Will work with Plebanski formulation of GR:

- Jerzy Plebanski, "On the separation of Einsteinian substructures", *J. Math. Phys.* **Vol. 18** 2511-2520 (1977).

Simple dimensional analysis shows that the action is incomplete: counterterms (quantum corrections) must be added.

A detailed analysis shows that the counterterms are of a very special type, and can all be combined into a single function ϕ of two complex variables.

This makes the behavior of gravity under renormalization much more transparent than in the usual metric-based treatment: renormalization group flow is one in the space of ϕ. Complete understanding may be within reach, e.g. testing of the asymptotic safety conjecture of Weinberg.

Part I: Polynomial formulations of gravity

Einstein formulated GR as the theory of the metric of spacetime

$$S_{EH}[g] = \frac{1}{16\pi G}\int_M d^4x\sqrt{-\det(g)}(R-2\Lambda). \tag{1}$$

The Newton constant G is absorbed into the fluctuating part of the metric field, giving it the mass dimension 1. Coupling constant - $\sqrt{G}$, mass dimension $[\sqrt{G}]=-1$: non-renormalizable! Hence, do not know its UV completion.

S-matrix is finite at one-loop (one-loop divergences become zero on-shell), but is divergent at two loops.

First order formalism

Einstein gravity as the second-order formalism: second derivatives in the action. First order formalism available. In its Einstein-Cartan form:

$$S[\theta,\omega] = \frac{1}{8\pi G}\int_M \epsilon^{IJKL}\theta^I \wedge \theta^J \wedge F_\omega^{KL} + \frac{\Lambda}{2}\epsilon^{IJKL}\theta^I \wedge \theta^J \wedge \theta^K \wedge \theta^L. \qquad (2)$$

Here θ^I are the frame field one-forms, $F_\omega = d\omega + (1/2)\omega\wedge\omega$ is the curvature of the spin connection, indices $I,\dots,K = 0,\dots,3$ are the internal ones, and ϵ^{IJKL} is the totally anti-symmetric tensor in the internal space.

Newton's constant

Newton's constant can be absorbed into the fields so that there is no dimensionfull coupling constant left: $\theta/\sqrt{G} = \tilde{\theta}, \Lambda G = \tilde{\Lambda}$. New mass dimensions:

$$[\tilde{\theta}] = 1, \quad [\omega] = 1, \quad [\tilde{\Lambda}] = 0. \tag{3}$$

Note: the presence of G in the usual metric-based perturbation theory (with background $\eta_{\mu\nu}$) is due to the split $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}h_{\mu\nu}$. No Newton's constant in pure gravity unless there is a background.

Quantization?

Starting point for quantization. Works in 3D, where gravity is shown to be (super) renormalizable in this formulation.

Does not work in higher D: no kinetic term.

Note: “renormalizable” action as there is only a very few terms that can be added to it compatible with mass dimensions and symmetries. In spite of this can't be quantized by the usual perturbative methods.

Deser, McCarthy, Yang work of 1989

Used the Palatini version of the first order formulation, $S = S[g_{\mu\nu}, \Gamma^{\rho}_{\mu\nu}]$. Kinetic term.

Realized that there is a "mismatch between the symmetries of its quadratic and cubic terms, which makes this ostensibly renormalizable system ill-defined about zero vacuum, and forces the usual expansion of the metric about a background".

Lesson: to start off perturnation theory one is forced to expand around a constant background - this is when dimensionful Newton constant appears, and this is how the theory becomes non-renormalizable.

Plebanski formulation

Why Plebanski: there is a "kinetic term".

Separation of metric, connection A and curvature Ψ from each other.

We will use the original self-dual version (version without the self-dual split is available).

$$S[B, A, \Psi] = \frac{1}{2\pi i G} \int_M B^a F^a_A + \frac{1}{2}(\Lambda \delta^{ab} + \Psi^{ab}) B^a B^b. \tag{4}$$

Here a, b are the $\mathfrak{su}(2)$ Lie algebra indices, Ψ^{ab} is a field that on-shell becomes the Weyl part of the curvature (it is required to be symmetric traceless), B^a is a Lie algebra valued 2-form field that on-shell becomes expressed through a tetrad.

Euler-Lagrange equations:

$$B^a B^b = \frac{1}{3}\delta^{ab}\delta^{cd}B^c B^d, \quad F_A^a = -(\Lambda\delta^{ab} + \Psi^{ab})B^b, \quad d_A B^a = 0. \tag{5}$$

First implies that B^a is the self-dual part of the two form $B^{IJ} := (1/2)\theta^{[K}\theta^{L]}$ for some tetrad θ^I, second and third identifies Ψ^{ab} as the self-dual part of the Weyl curvature tensor.

Again, no dimensionfull coupling after field rescalings. There is now a "kinetic term" for the fields, except for Ψ. Let us treat Ψ as an external field (i.e. postpone integration over Ψ).

Compare e.g. QED: can start with fermions in an external electromagnetic field

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) + m)\psi. \tag{6}$$

Quantum corrections will generate the kinetic term for A: the usual F^2 Lagrangian. Let us see if anything like this happens for Plebanski theory.

Another example: Gross-Neveau model

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi + \sigma\overline{\psi}\psi - \frac{\sigma^2}{2g^2}. \qquad (7)$$

2D theory, mass dimensions $[\psi] = 1/2, [\sigma] = 1$. If no last term - a trivial theory (free left or right movers). The σ^2 term gets generated by quantum corrections. After it is integrated out - interacting $(\overline{\psi}\psi)^2$, asymptotically free theory.

Somewhat similar phenomenon happens in Plebanski theory.

Part II: Possible counterterms

The first step in analyzing the UV behavior is to produce a list of possible divergent terms (of mass dimension four compatible with symmetries).

After G is absorbed into the fields, the mass dimensions are:

$$[A] = 1, \quad [B] = 2, \quad [\Psi] = 0. \tag{8}$$

It is thus obvious that all powers of Ψ will appear.

Thus, in addition to the term $\Psi^{ab}B^aB^b$ need to add the terms of the form

$$\frac{1}{2}(\Psi^{k_1})^{ab}(\mathrm{Tr}(\Psi^2))^{k_2}\ldots(\mathrm{Tr}(\Psi^n))^{k_n}B^aB^b \tag{9}$$

The theory does seem to be as non-renormalizable as in the usual perturbative quantum gravity. Usual case: dimensionfull Newton's constant; our case - a field Ψ of mass dimension zero.

Other terms

Clear that all powers of Ψ will get generated also in front of BF and FF terms. The renormalized action with all these counterterms is:

$$i\mathcal{L} = \frac{1}{2}\bar{X}(\Psi)^{ab}F_A^aF_A^b + \bar{Y}(\Psi)^{ab}B^aF_A^b + \frac{1}{2}\bar{Z}(\Psi)^{ab}B^aB^b, \qquad (10)$$

where $\bar{X}(\Psi), \bar{Y}(\Psi), \bar{Z}(\Psi)$ are all tensors, polynomials in Ψ and its traces. The coefficients of these polynomials are undetermined. Infinite number of them, seemingly no predictive power. Usual non-renormalizability? Not quite.

Field B redefinition

Can redefine the field $B \to B + H(\Psi)F(A)$ to get rid of the $F^a F^b$ term. Can then "rescale" the field B to map the $B^a F^b$ term into its canonical form. After this B field redefinitions one gets

$$i\mathcal{L} = \tilde{B}^a F_A^a + \frac{1}{2}\tilde{\Psi}(\Psi)^{ab}\tilde{B}^a\tilde{B}^b, \tag{11}$$

where

$$\tilde{\Psi}(\Psi) = (Y(\Psi)^T Z(\Psi)^{-1} Y(\Psi) - X(\Psi))^{-1}. \tag{12}$$

Field Ψ redefinition

The whole effect of the counterterms is to replace the curvature field Ψ^{ab} by a non-trivial, depending on many new parameters (coupling constants) functional $\tilde{\Psi}^{ab}(\Psi)$.

Rewrite:

$$\tilde{\Psi}(\Psi)^{ab} = \Phi^{ab}(\Psi) + \delta^{ab}\phi(\Psi), \qquad (13)$$

where $\Phi^{ab}(\Psi)$ is (a multiple of) the traceless part of $\tilde{\Psi}$. The field Φ^{ab} just replaces the original field Ψ^{ab} after the renormalization!

Renormalized action

The effect of counterterms is in replacing the bare curvature field Ψ by the renormalized one Φ, and in appearance in the action of a new "trace" term:

$$\frac{1}{i}\int_M B^a F_A^a + \frac{1}{2}\left(\Lambda\delta^{ab} + \Phi^{ab} + \delta^{ab}\phi(\Phi)\right) B^a B^b, \qquad (14)$$

Still non-renormalizable in the strict sense of the word (as still an infinite number of undetermined constants).

More on the function ϕ

Being a scalar function, ϕ can only be a function of eigenvalues of Ψ^{ab}, of which there are two (traceless!)

$$\phi = \phi(\lambda_1, \lambda_2). \tag{15}$$

Renormalization group flow is that in the space of such functions!

Modifications of gravity?

The metricity equations are modified to

$$B^a B^b + \frac{d\phi(\Phi)}{d\Phi_{ab}}(B^c B_c) = \frac{1}{3}\delta^{ab}(B^c B_c). \tag{16}$$

This equation no longer implies that the two-form field B^a is metric. We see that non-metricity is unavoidable whenever there is non-zero "curvature" Φ.

Non-metric gravity

As a detailed analysis shows, certain metric g can still be introduced. Equations of motion one gets are second order in derivatives relating g and derivatives of g with components of Ψ and derivatives of Ψ. Schematically

$$D^2 g + f(g,\Psi)\tilde{D}^2\Psi = \Psi + \phi(\Psi). \tag{17}$$

Can be solved for Ψ, generating an infinite series in derivatives of g.

Quantum modified Plebanski theory (="non-metric" gravity) equivalent to the usual quantum modified GR with an infinite number of higher derivative terms added to the action.

Comparing the two expansions

Expansion in powers of Ψ is not the same as the expansion in powers of $R(g)$! Even $\phi = \mathrm{Tr}(\Psi)^2$ gives an infinite expansion when interpreted in metric terms!

Good, because the theory modified by a finite number of $R(g)^n$ corrections is plagued with ghosts. Plebanski formulation gives a way to avoid ghosts by keeping equations second order.

Other counterterms

However, other counterterms may be possible, of high power in Ψ, e.g.

$$\Psi^{aa_1}(d_A\Psi)^{a_1a_2}(d_A\Psi)^{a_2a_3}(d_A\Psi)^{a_3a_4}(d_A\Psi)^{a_4a}, \tag{18}$$

$$f^{abc}\Psi^{ba_1}(d_A\Psi)^{a_1a_2}(d_A\Psi)^{a_2c}F^a, \tag{19}$$

as well as terms similar to the last one with B^a instead of F^a.

Don't have to worry about these for small curvatures, but important for understanding renormalizability properties. This is why have to study details of quantum theory.

Part III: Quantum Theory

Would like to verify which of the counterterms do appear, and - eventually - compute the beta-function for the function ϕ.

Problem: Even with Ψ non-fluctuating, there is still a mismatch between the symmetries of the quadratic and cubic terms.

This is what "killed" the Deser et all attempt at quantization in Palatini formulation.

Choose a constant background - not learn anything new.

A possible solution

Will use a trick due to Stueckelberg, which is to introduce an extra field so that the full action acquires the desired symmetry (that of the kinetic term). The symmetry can then be gauge fixed.

Action with the Stueckelberg field

Convenient to define: $\Lambda^{ab} = \Lambda\delta^{ab} + \Psi^{ab}$, and rescale B^a by Λ^{ab}. The original action then becomes:

$$i\mathcal{L} = (\Lambda^{-1})^{ab}\left(B^a F^b + \frac{1}{2}B^a B^b\right). \tag{20}$$

We introduce a new field η - Lie algebra valued one form. The new action is:

$$i\mathcal{L} = \frac{1}{2}(\Lambda^{-1})^{ab}\left((B + F_{A+\eta})^a (B + F_{A+\eta})^b - F_A^a F_A^b\right). \tag{21}$$

Reduces to the original action when $\eta = 0$.

Symmetries

The (topological) symmetry of the kinetic term becomes that of the full action:

$$\eta \to \eta + \tau, \qquad B \to B - d_{A+\eta}\tau - (1/2)[\tau, \tau]. \tag{22}$$

This symmetry is sufficient to set $\eta = 0$. Another, more interesting gauge is the *self-dual* one given by $B^- = 0$.

External fields

Even after the gauge fixing there is no kinetic term for Ψ and A. Could again try to choose a constant background for A. More interesting option is to keep A classical as well.

Interpretation: fixing a background for the Ψ and the connection A, and integrating over the fluctuation η of the connection and over the "geometrical" field B. Quantum gravity in the background of A, Ψ.

Honest quantum theory without need for a constant background. Price: external fields. What are the physical questions that can be asked in this theory?

Integrating out B

After the gauge-fixing (self-dual), the result of integration over B is:

$$\mathcal{L} = -\frac{1}{2}(\Lambda^{-1})^{ab}(F^{+}_{A+\eta})^{\mu\nu a}(F^{+}_{A+\eta})^{a}_{\mu\nu}, \tag{23}$$

plus a term that only depends on the background, plus a set of gauge-fixing terms for η (essentially the same as in Donaldson theory).

Integrating out η

We have only performed the computation to one loop order. However, the theory is simple enough - not much more complicated then YM. Higher loops are also not hard. Results:

- No divergences containing derivatives of Ψ appear. The counterterm corrected Lagrangian is then

$$\mathcal{L} = -\frac{1}{2}(\tilde{\Lambda}(\Lambda^{-1}))^{ab}(F_A^+)^{\mu\nu a}(F_A^+)^a_{\mu\nu}, \tag{24}$$

plus the background fields term of the original action.

Preliminary results on the beta-functions

$$\frac{d\alpha}{d\log\mu} = -4C_2\alpha^2, \qquad \frac{d\tilde{g}}{d\log\mu} = -16C_2\alpha\tilde{g}, \tag{25}$$

where C_2 is the quadratic Casimir in the fundamental representation ($C_2 = 2$ in our case), and $\alpha := g^2/(4\pi)^2$, where g is the gauge field coupling. Asymptotically free first couplings! Interpretation? Will become clear when the full beta function for ϕ is computed.

Summary

New approach to perturbative quantum gravity, which replaces the expansion in powers of $R(g)$ by another expansion in powers of the "Lagrange multiplier" field Ψ.

Very non-trivial relation between the two. The quantum corrected theory is always second order in derivatives, something not possible with any finite number of $R^n(g)$ terms.

The beta-function is a functional of $\phi(\lambda_1, \lambda_2)$. Its computation seems within reach.

Open questions

- Is asymptotic safety conjecture of Weinberg correct? I.e. is there a non-trivial limit $\phi^* = \lim_{\mu\to\infty} \phi_\mu$?
- Interpretation of the quantum theory in the external field. Gravitons (weak field limit of the theory) are understood. Can one extract their scattering aplitudes from the effective action for A, Ψ? What physical questions can be asked?
- Dependence on the gauge?
- Higher loops?
- Lots of interesting questions about the classical theory (quantum corrected gravity) = non-metric gravity. Work in progress.