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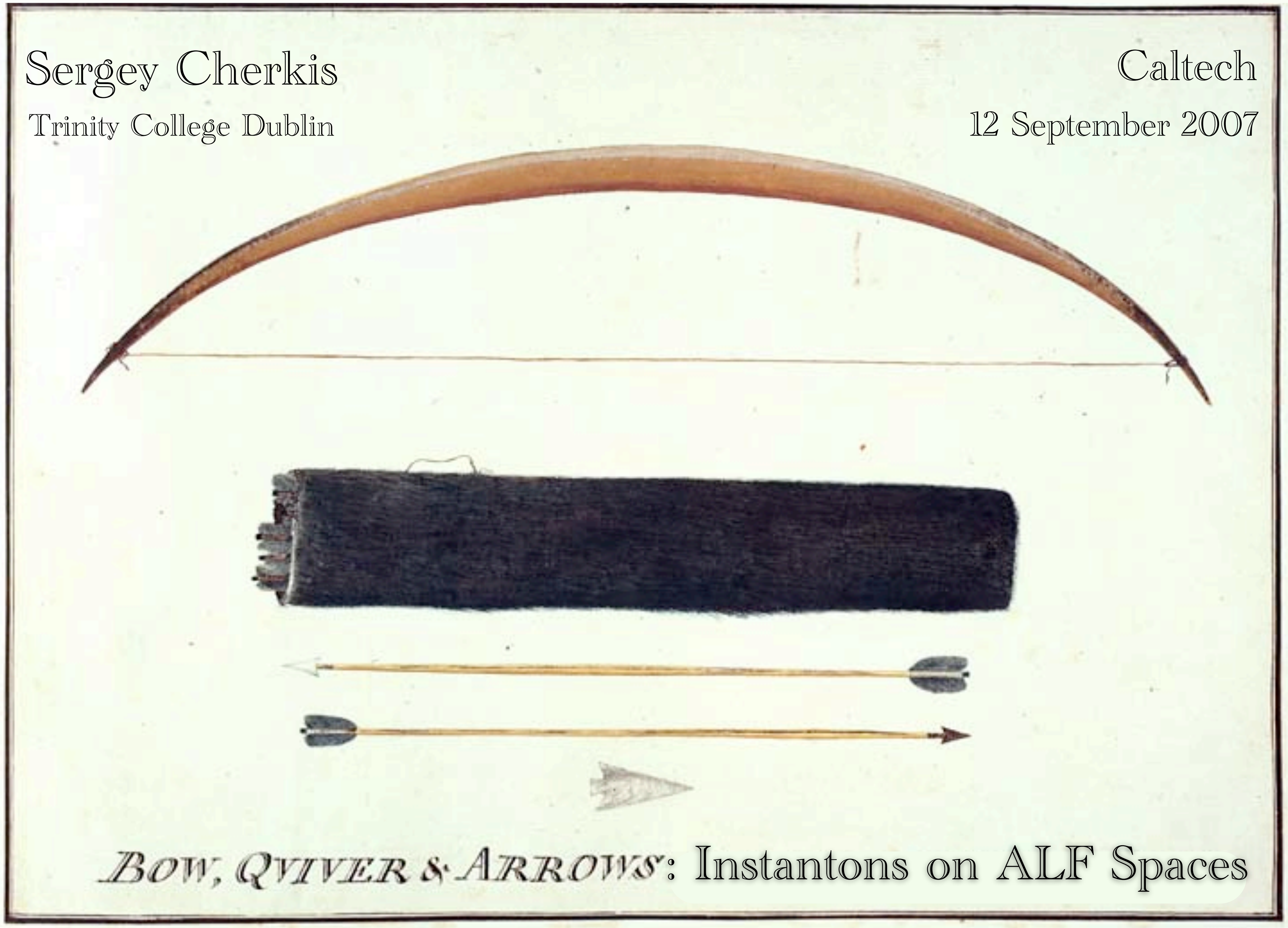


# BOW, QVIVER & ARROWS: Instantons on ALF Spaces

From: http://www.captcook-ne.co.uk/ccne/themes/objects.htm

# Self-dual Gravitational Instantons

are complete four-dimensional Riemannian manifolds that satisfy one of the following equivalent conditions:

i. hyperkähler
ii. admits covariantly constant spinors
iii. Calabi-Yau two-fold
iv. preserves 1/2 Supersymmetry
v. self-dual curvature form

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta}$$

Distinguished by

- Asymptotic behaviour: K3
- Topology: A, D, E, etc.

<u>Questions</u>:

Classification, metrics, <u>Yang-Mills Instantons</u>

# Conjecture:

Any gravitational instanton metric with finite Pontrjagin number asymptotically approaches a metric with a local triholomorphic isometry.

$$\int_M R \wedge R < \infty \qquad \Longrightarrow \qquad ds^2 \xrightarrow[|\vec{x}| \to \infty]{} V^{-1}(d\theta + \omega)^2 + V d\vec{x}^2$$

| | |
|---|---|
| ALE | $V = \dfrac{1}{\lvert\vec{x}\rvert}$ |
| ALF $\quad$ $A_k$ and $D_k$ | $V = C + \dfrac{1}{\lvert\vec{x}\rvert}$ |
| ALG | $V = C + \dfrac{N}{2}\log(x_1^2 + x_2^2)$ |
| ALH | $V = C_1 + C_2 x_1$ |

# The Taub-NUT Space

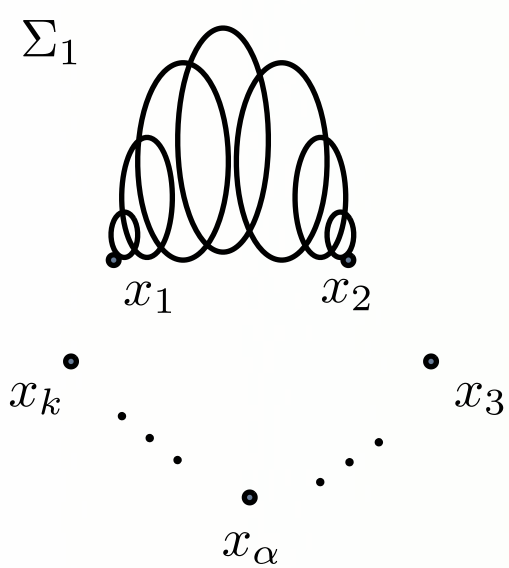
$$ds^2 = V^{-1}\left(d\theta + \omega\right)^2 + V d\vec{x}^2,$$

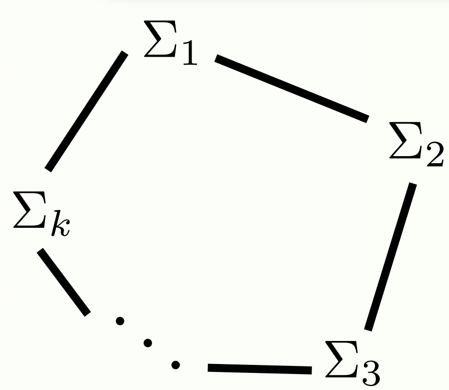
$$d\omega = *_3 dV, \theta \sim \theta + 4\pi, \qquad V = l + \frac{1}{|\vec{x}|}$$

Self-dual Abelian connection:

$$a_0 = \frac{s}{4\pi}\frac{d\theta + \omega}{V}$$

# Multi-Taub-NUT (Ak ALF)




$$V = l + \sum_{\alpha=1}^{k+1} \frac{1}{|\vec{x} - \vec{x}_\alpha|}$$

Self-dual Abelian connections:

$$a_\alpha = \frac{1}{4\pi}\left( (V_\alpha - V_{\alpha+1})\frac{d\theta + \omega}{V} + \omega_\alpha - \omega_{\alpha+1} \right) \qquad V_\alpha = \frac{1}{|\vec{x} - \vec{x}_\alpha|}$$

$$a_0 = \frac{s}{4\pi}\frac{d\theta + \omega}{V} \qquad d\omega_\alpha = *dV_\alpha$$

# Instantons on ALF Spaces

$$F = *F$$

Action $S = \int F \wedge *F$ is finite

Monodromy at infinity $\left(\frac{\partial}{\partial\theta} - iA_\theta\right) W(\vec{x},\theta) = 0,\; W(\vec{x},0) = 1 \qquad W = \lim_{x\to\infty} W(\vec{x},4\pi)$

## •Maximal Symmetry Breaking:

EigenValues of W are distinct $-\frac{\pi}{l} < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \frac{\pi}{l}$

EigenBundles of W are line bundles $\mathcal{L}_i \to S^2_\infty$ with Chern classes $j_i$

Monopole Charges: if $M=\min(j_1, j_1+j_2,\ldots, j_1+j_2+\ldots+j_n)$

then the monopole charges are $(m_1,m_2,\ldots,m_n)=(j_1-M, j_1+j_2-M,\ldots, j_1+j_2+\ldots+j_n-M)$

## Instanton Number:

$$n = \frac{1}{32\pi^2}\int \mathrm{Tr}\; F\wedge F - (m_1(l\lambda_1+\pi) + m_2 l(\lambda_2-\lambda_1) + \ldots m_n(\pi - l\lambda_n))$$

Question: Find explicit SD connections on ALF spaces

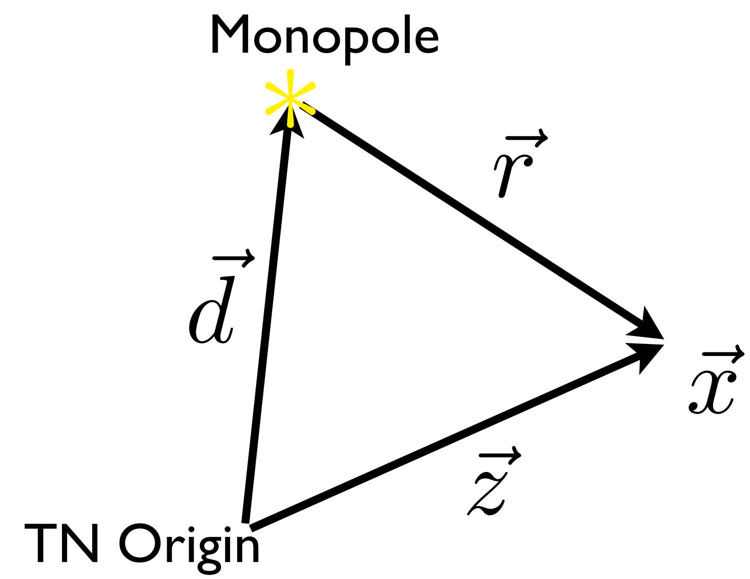
# Explicit Solution on TN: m=1, n=0

SCh & Brian Durcan

$$V = l + \frac{1}{2z},$$

$$a = z + d$$

$$\mathcal{D} = (z+d)^2 - r^2$$

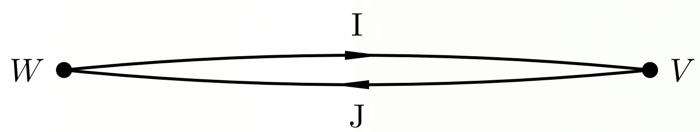



$$\begin{aligned}
\mathcal{K} &= (a^2 + r^2)\,\cosh(2\lambda r) + 2ra\,\sinh(2\lambda r) \\
\mathcal{L} &= (a^2 + r^2)\,\sinh(2\lambda r) + 2ra\,\cosh(2\lambda r)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A} \;=\; & \frac{1}{\mathcal{L}}\Big\{ d\vec{x}\cdot(\vec{\sigma}\times\vec{r})\Big(\frac{(2\lambda r - \sinh(2\lambda r))\mathcal{D}}{2r^2} - \sinh(2\lambda r)\left(1 + \frac{a}{r}\tanh(\lambda r)\right)\Big) \\
+ \; & \frac{d\vec{x}\cdot(\vec{r}\times\vec{z})}{z}\frac{\vec{\sigma}\cdot\vec{r}}{r}\left(1 - \frac{\mathcal{K}}{\mathcal{D}}\right) - \frac{r}{z}d\vec{x}\cdot(\vec{\sigma}\times\vec{z}) \\
+ \; & \frac{d\theta - \omega}{V}\Big(\frac{\vec{\sigma}\cdot\vec{r}}{r}\left((\lambda + \frac{1}{2z})\mathcal{K} - \frac{\mathcal{L}}{2r}\right) - \frac{r}{z}\vec{\sigma}\cdot\vec{d}_\perp\Big)\Big\}
\end{aligned}$$

**Next Question:** Find explicit m=0, n=1 SD connections on TN

# Ingredients 1: Arrows and Limbs




$V = \mathbb{C}^v$ and $W = \mathbb{C}^w$

$$g_v : (I, J) \mapsto (g_v^{-1} I, J g_v) \qquad\qquad g_w : (I, J) \mapsto (I g_w, g_w^{-1} J)$$

## Moment maps:

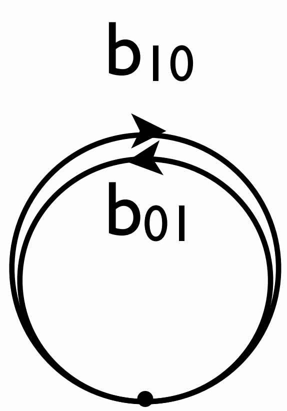
$$\mu_V^{\mathbb{C}} = \mu_V^1 + i\mu_V^2 = IJ, \quad \mu_V^{\mathbb{R}} = \mu_V^3 = \frac{1}{2}(J^\dagger J - I I^\dagger), \qquad\qquad \mu_W^{\mathbb{C}} = \mu_W^1 + i\mu_W^2 = -JI, \quad \mu_W^{\mathbb{R}} = \mu_W^3 = \frac{1}{2}(I^\dagger I - J J^\dagger).$$

A convenient way of writing the moment maps is

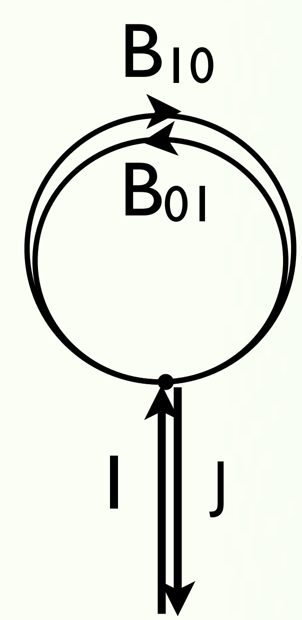
$$Q_V = \begin{pmatrix} J^\dagger \\ I \end{pmatrix} \qquad \boldsymbol{\mu}_V = \mu_V^i \sigma_i = \mathrm{Vec}(Q_V Q_V^\dagger) \qquad \mathrm{Vec}(M^0 + M^j \sigma_j) = M^j \sigma_j$$

# Example: ADHM

$\mathbb{R}^4$

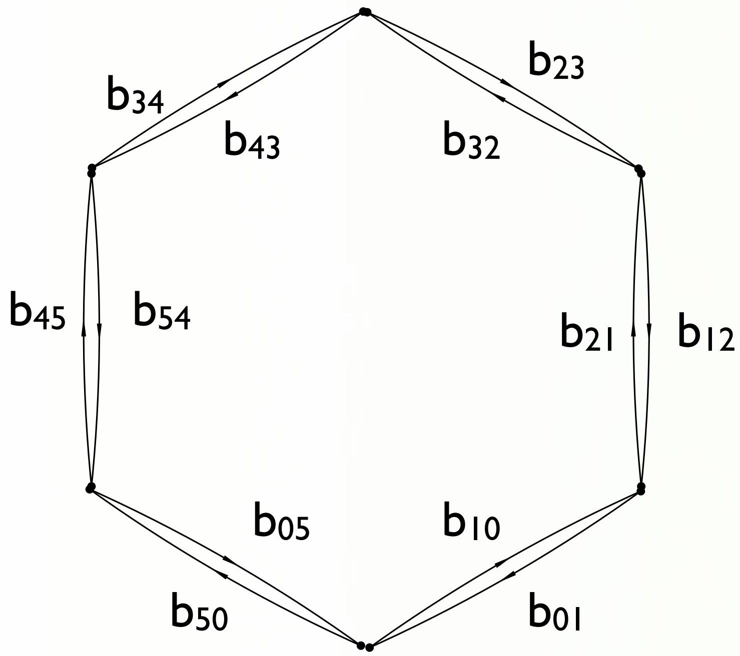



Instanton Data:



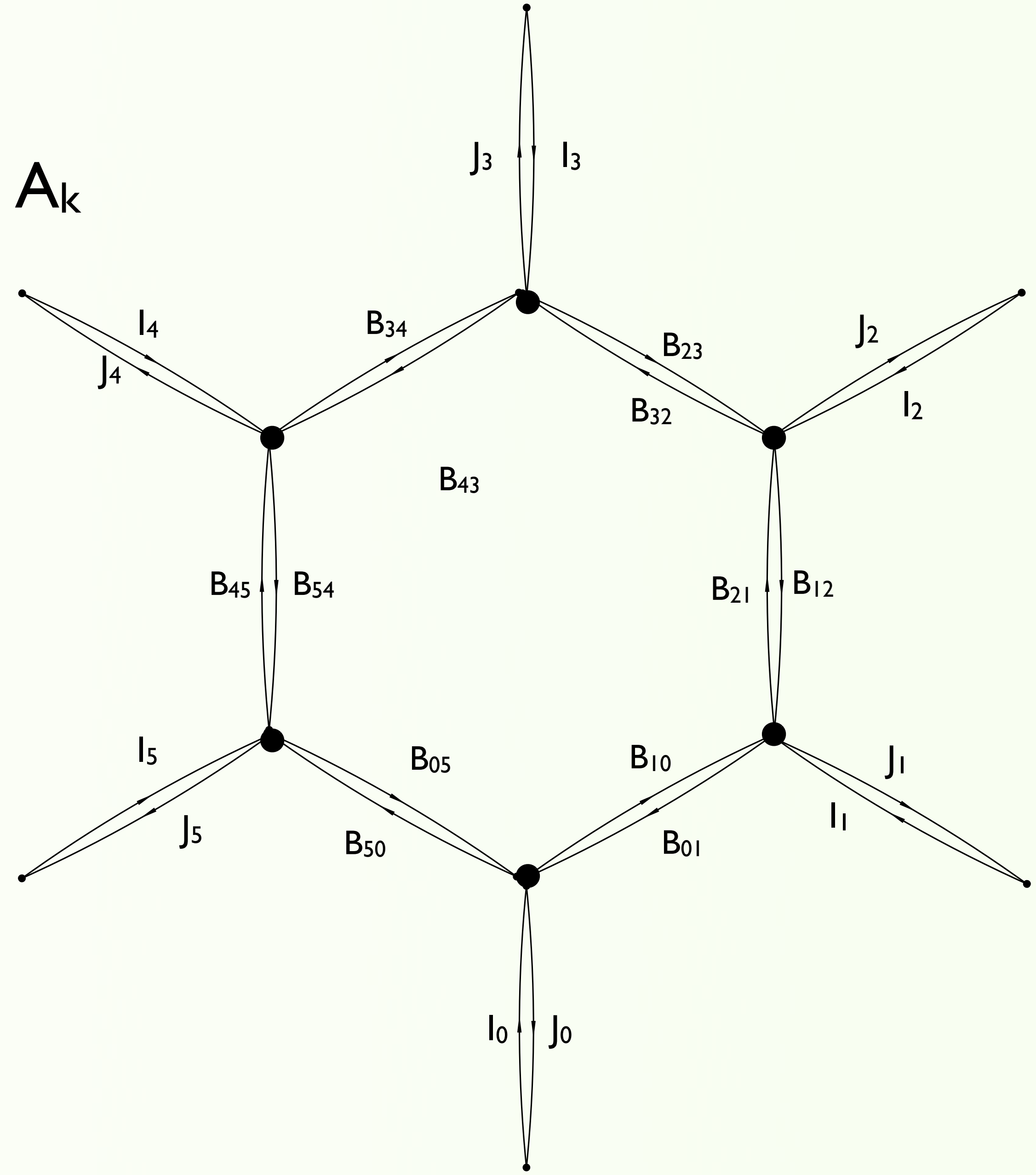
# Example: Kronheimer & Nakajima (Instantons on $\widetilde{\mathbb{R}^4/\Gamma}$)

$A_k$ ALE: $\widetilde{\mathbb{R}^4/\mathbb{Z}_{k+1}}$




Affine Dynkin diagram

Instantons on $A_k$ ALE




Moment maps at $V_1$

$\mu^C = B_{10} B_{01} - B_{12} B_{21} + I_1 J_1$

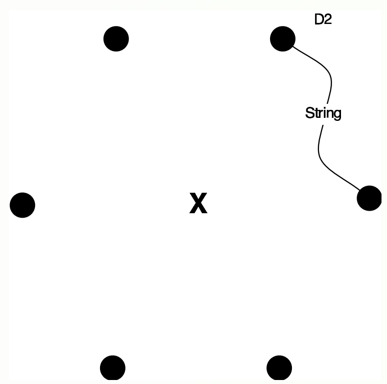
$\mu^R = B^+_{01} B_{01} - B_{10} B^+_{10} + B_{12} B^+_{12} - B^+_{21} B_{21} + I_1 I^+_1 - J^+_1 J_1$

# ALE Spaces:

Douglas & Moore
Johnson & Myers

## Kronheimer Construction from String Theory

D2-brane on $\mathbb{R}^4/\Gamma \times \mathbb{R}^6$




$r = |\Gamma|$ rank of $\Gamma$

Super Yang-Mills with gauge group $U(r)$

Equivariance conditions

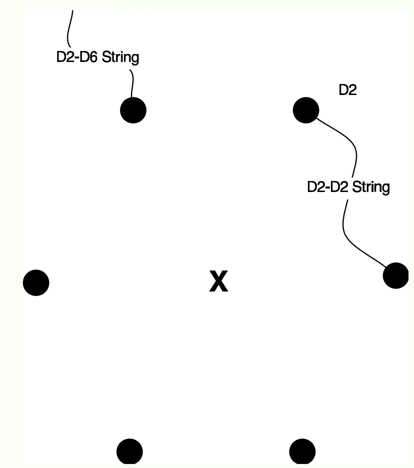
$$\begin{aligned} A_\mu &= \gamma^{-1}(g) A_\mu \gamma(g) \\ \phi^p &= \gamma^{-1}(g) \phi^p \gamma(g) \\ \Phi^I &= R(g)^I_J \gamma^{-1}(g) \Phi^J \gamma(g) \end{aligned}$$

$\gamma$ is an $r$-dimensional representation of $\Gamma$,

$R$ is a two-dimensional representation of $\Gamma$.

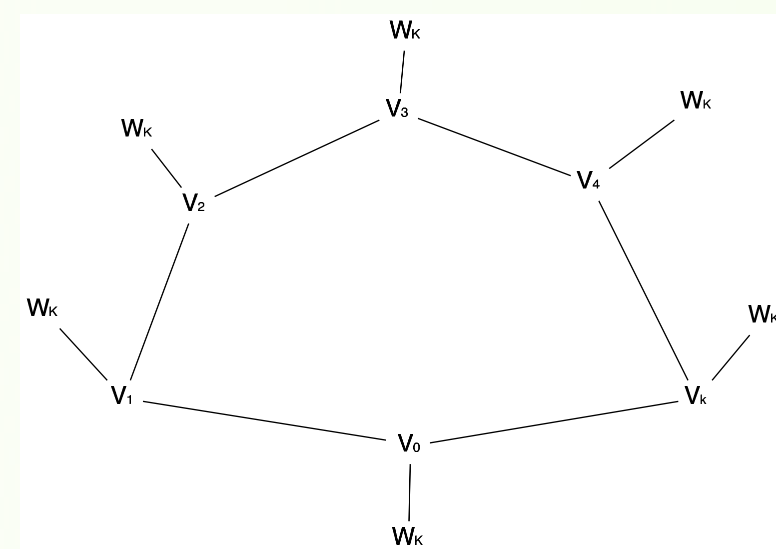
## Kronheimer-Nakajima Construction from String Theory

$N$ instantons in $U(K)$ on ALE space




N D2-branes and K D6-branes on $\mathbb{R}^4/\Gamma \times \mathbb{R}^6$

Super Yang-Mills with gauge group $U(r)$ and $K$ scalar fields in the defining representation.



# Ingredients 2: "Strings"

U(n)

$$(T_0(s), T_1(s), T_2(s), T_3(s)).$$

$$g(s) : \begin{pmatrix} T_0(s) \\ T_1(s) \\ T_2(s) \\ T_3(s) \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}T_0 g + i g^{-1}\frac{d}{ds} g \\ g^{-1}T_1 g \\ g^{-1}T_2 g \\ g^{-1}T_3 g \end{pmatrix}$$

$$\begin{aligned} \mu^1 &= \frac{d}{ds}T_1 - i[T_0, T_1] + i[T_2, T_3], \\ \mu^2 &= \frac{d}{ds}T_2 - i[T_0, T_2] + i[T_3, T_1], \\ \mu^3 &= \frac{d}{ds}T_3 - i[T_0, T_3] + i[T_1, T_2]. \end{aligned}$$

## Convenient Notation:

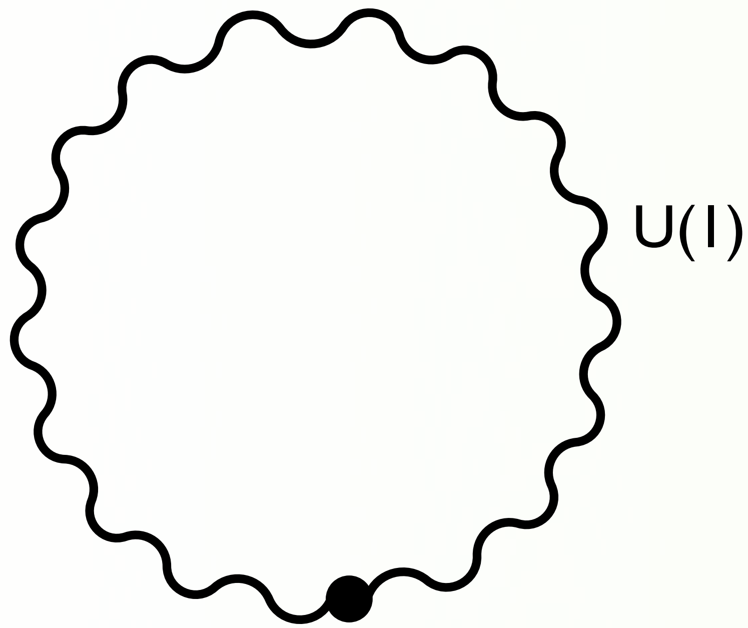
$$\not{T} = \sigma_1 \otimes T_1 + \sigma_2 \otimes T_2 + \sigma_3 \otimes T_3$$

$$\not{\mu} = [\tfrac{d}{ds} - iT_0, \not{T}] + \mathrm{Vec}(\not{T}, \not{T}).$$

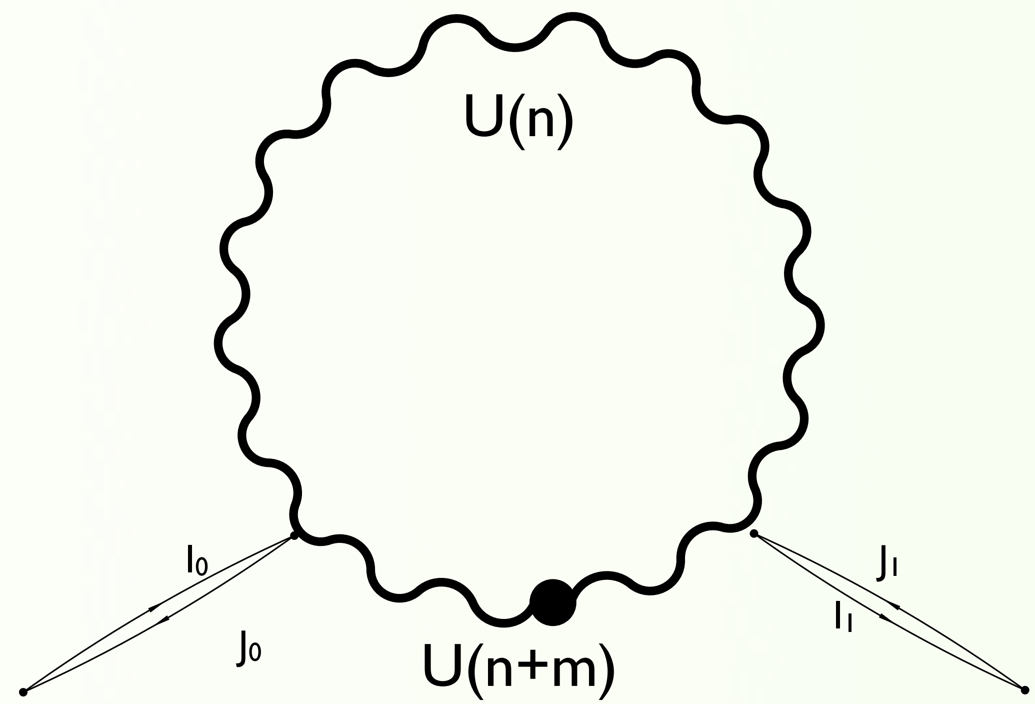
# Example: Calorons

Nahm

$\mathbb{R}^3 \times \mathbf{S}^1$



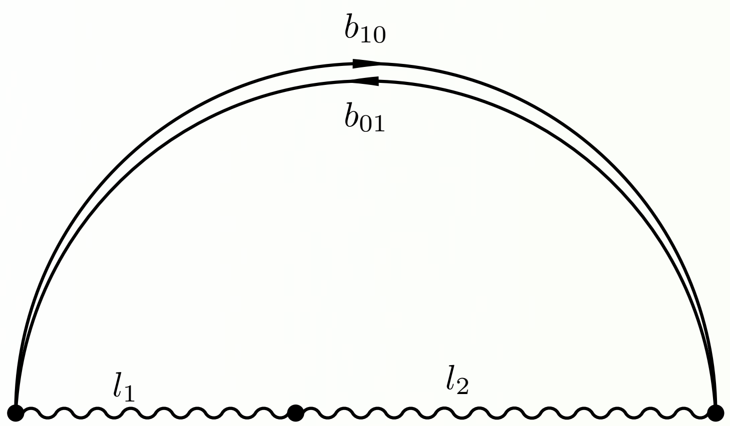

Instantons on $\mathbb{R}^3 \times \mathbf{S}^1$




String Theory derivation via Chalmers-Hanany-Witten configuration

Diaconescu

# Taub-NUT Bow Diagram




$$\begin{pmatrix} t_0 \\ t_j \\ b_{01} \\ b_{10} \end{pmatrix} \mapsto \begin{pmatrix} h^{-1}t_0h + ih^{-1}\frac{d}{ds}h \\ h^{-1}t_jh \\ h^{-1}(-\frac{l}{2})b_{01}h(\frac{l}{2}) \\ h^{-1}(\frac{l}{2})b_{10}h(-\frac{l}{2}) \end{pmatrix}$$

$$t = t_1 + it_2 \text{ and } \mathrm{D} = d/ds - it_0 - t_3$$

**Moment maps:**

$$[\mathrm{D}, t] - \delta(s+\tfrac{l}{2})b_{01}b_{10} + \delta(s-\tfrac{l}{2})b_{10}b_{01} = 0,$$

$$[\mathrm{D}^\dagger, \mathrm{D}] + [t^\dagger, t] + \delta(s+\tfrac{l}{2})(b_{10}^\dagger b_{10} - b_{01}b_{01}^\dagger) + \delta(s-\tfrac{l}{2})(b_{01}^\dagger b_{01} - b_{10}b_{10}^\dagger) = 0$$

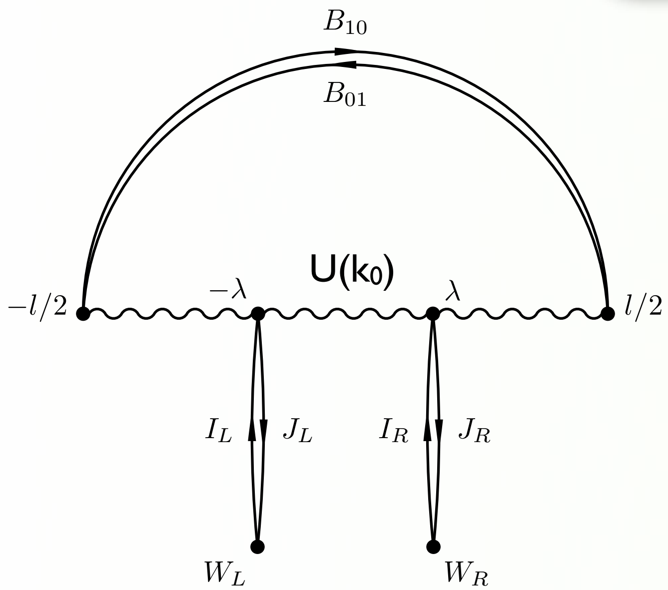
$$\boxed{ds^2 = \frac{1}{4}\left[(l+\frac{1}{r})d\vec{r}^2 + \frac{1}{l+1/r}(d\tau+\omega)^2\right]} + l_1\frac{l+1/r}{l_1+1/r}\left[dt_0 + \frac{d\tau+\omega}{2(l+1/r)}\right]^2.$$

**Metric**

$$\phi = -l_1 t_0.$$

**Natural Connection:** $\boxed{a_0 = \frac{l_1}{2}(d\tau+\omega)/(l+1/r).}$

# $k_0$ Instantons on Taub-NUT




- a rank $k_0$ vector bundle $E \to [-l/2, l/2]$ with the Nahm data $(T_0, \vec{T})$ on the intervals $[-l/2, -\lambda], [-\lambda, \lambda]$, and $[\lambda, l/2]$ (we do not presume continuity at $s = \pm\lambda$),
- linear maps $B_{10} : E_{-l/2} \to E_{l/2}$ and $B_{01} : E_{l/2} \to E_{-l/2}$,
- linear maps $I_L : W_L \to E_{-\lambda}$, $J_L : E_{-\lambda} \to W_L$, $I_R : W_R \to E_\lambda$, and $J_R : E_\lambda \to W_R$.

$$\begin{pmatrix} T_0 \\ T_j \\ B_{01} \\ B_{10} \\ I_\alpha \\ J_\alpha \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}(s) T_0 g(s) + i g^{-1}(s) \frac{d}{ds} g(s) \\ g^{-1}(s) T_j g(s) \\ g^{-1}(-\frac{l}{2}) B_{01} g(\frac{l}{2}) \\ g^{-1}(\frac{l}{2}) B_{10} g(-\frac{l}{2}) \\ g^{-1}(\lambda_\alpha) I_\alpha \\ J_\alpha g(\lambda_\alpha) \end{pmatrix}$$

Let

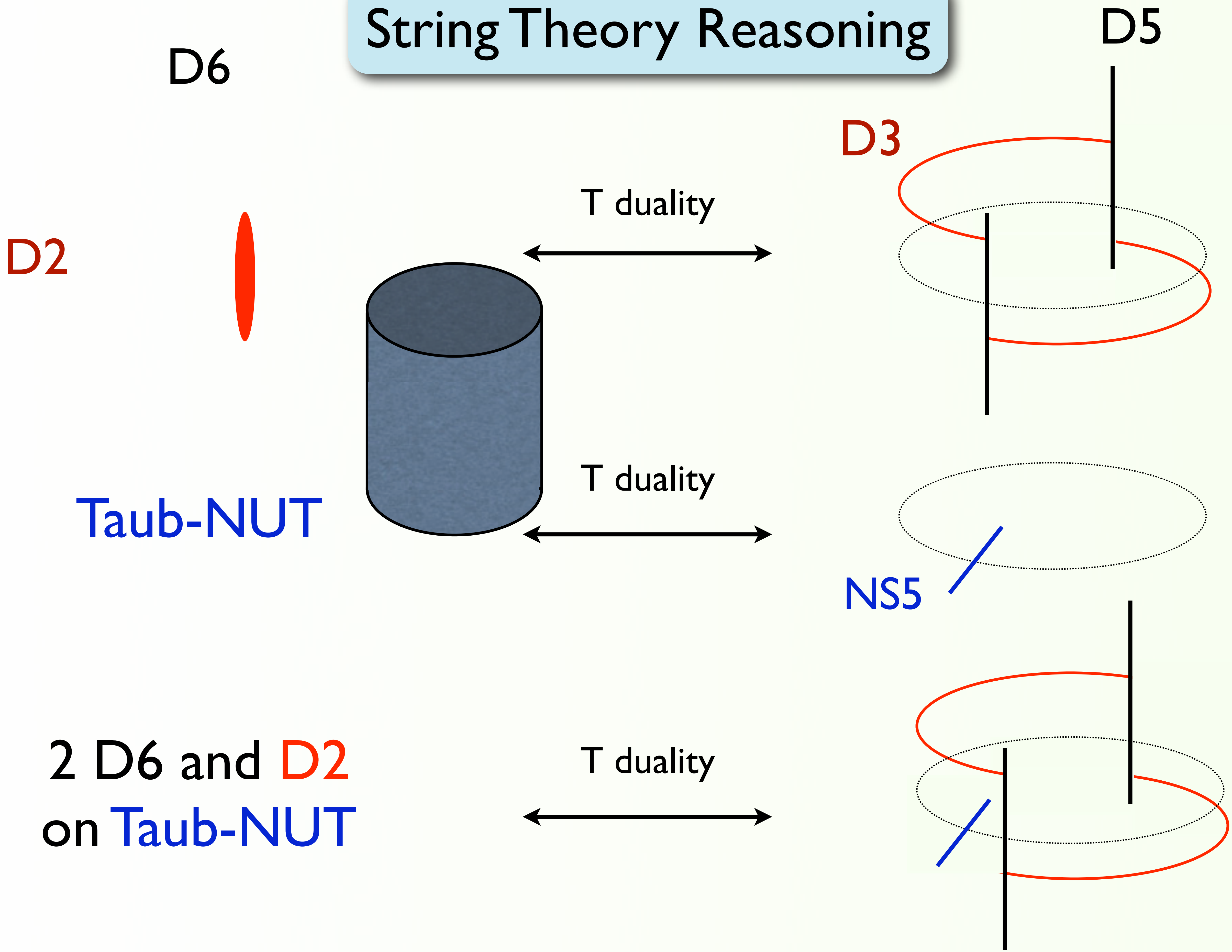
$$D = \frac{d}{ds} - iT_0 - T_3 \text{ and } T = T_1 + iT_2,$$

Moment map conditions:

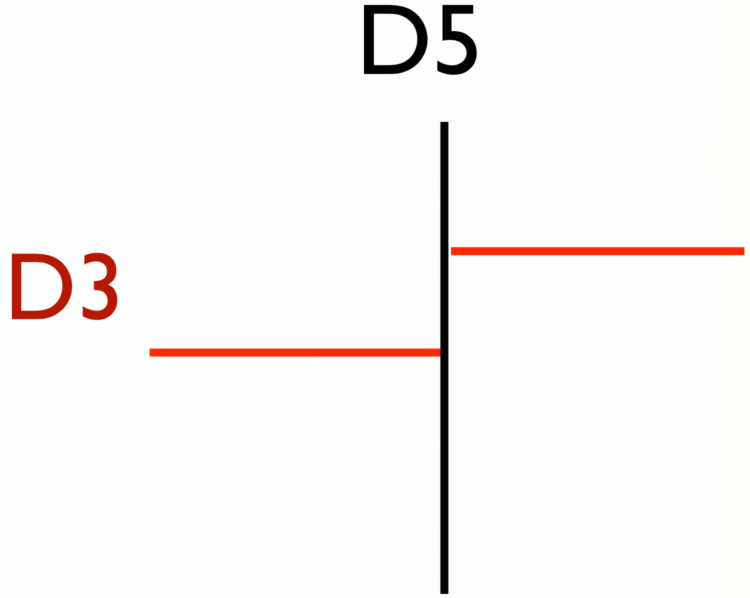
$$[D, T] - \delta(s+\tfrac{l}{2}) B_{01} B_{10} + \delta(s-\tfrac{l}{2}) B_{10} B_{01} + \sum_{\alpha \in \{L,R\}} \delta(s-\lambda_\alpha) I_\alpha J_\alpha = 0$$

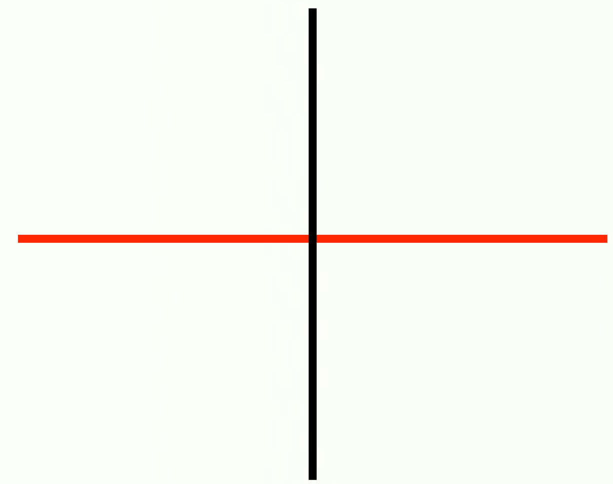
$$[D^\dagger, D] + [T^\dagger, T] + \delta(s+\tfrac{l}{2})(B_{10}^\dagger B_{10} - B_{01} B_{01}^\dagger) + \delta(s-\tfrac{l}{2})(B_{01}^\dagger B_{01} - B_{10} B_{10}^\dagger) + \\ + \sum_{\alpha \in \{L,R\}} \delta(s-\lambda_\alpha)(J_\alpha^\dagger J_\alpha - I_\alpha I_\alpha^\dagger) = 0.$$

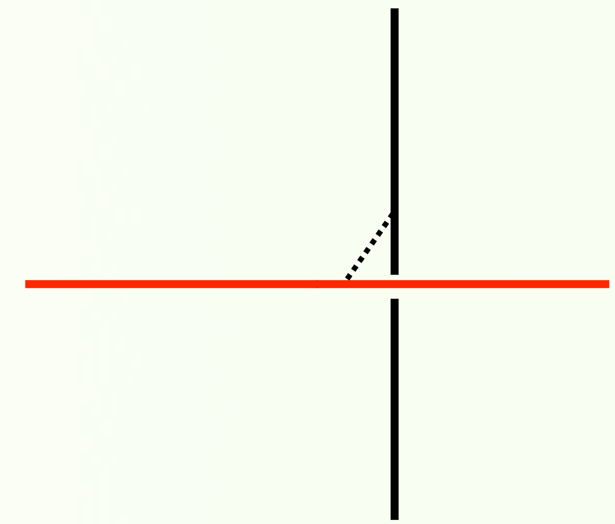
# String Theory Reasoning



# Gauge Theory on D3-brane

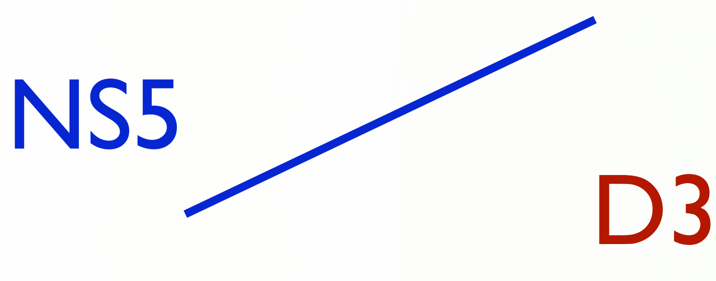


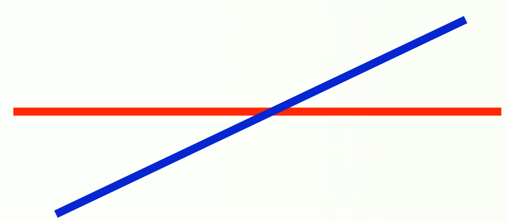



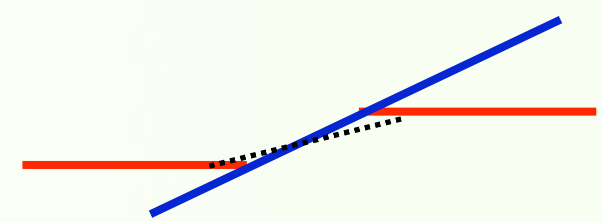


Massless fundamental hypermultiplet: f

Massive fundamental hypermultiplet from D3-D5 open string mode





Massless bifundamental hypemultiplet: B

Massive bifundamental hypemultiplet from D3-D3 open string mode

# Impurity Theory on D3

à la Kapustin & Sethi

## N=2, D=4 Yang-Mills with hyperplanes of impurities

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D5 | x | x | x | x | x | x | | | | |
| D3 | x | x | x | | | | x | | | |
| NS5 | x | x | x | | | | | x | x | x |
| Vector Multiplet | $A_0$ | $A_1$ | $A_2$ | | | | | $Y_1$ | $Y_3$ | $Y_3$ |
| Adjoint Hyper | | | | Im $H_1$ | Im $H_2$ | Re $H_2$ | Re $H_1$ | | | |

$\lambda_\alpha, \alpha = 1,2$ Majorana

$\psi$ Dirac

$\mu = 0,1,2 \qquad \alpha = 1,2 \qquad i,j = 1,2,3$

$$L = L_1 + L_2$$

$$L_1 = R_6 \int d^3x_\mu dx_6 \left\{ \frac{1}{2}|F_{\mu\nu}|^2 + \frac{1}{2}|D_\mu Y^i|^2 - \frac{1}{2}|D_6 Y^i|^2 - \frac{1}{2}\sum_{i<j} |[Y^i, Y^j]|^2 + \frac{1}{2}|D_\mu H^j|^2 - \sum_{ij} |[Y^i, H^j]|^2 \right\}$$

$$L_2 = l \int d^3x_\mu dx_6 \quad \left\{ \frac{1}{l}\left( \sum_p \delta(x_6 - \lambda_p) \left( |D_\mu f^p|^2 - |Y^i f^p|^2 \right) + \right.\right.$$

$$+ |D_\mu B|^2 + \delta(x_6)|Y^i(x_6+)B - BY^i(x_6-)|^2 \Big) +$$

$$+ \frac{1}{2}|\mathcal{D}|^2 + \mathrm{Tr} i\mathcal{D}^\alpha_\beta \left( [H_\alpha, H^{\dagger\beta}] + \frac{1}{l}\left[ \sum_p \delta(x_6 - \lambda_p) f^p_\alpha \otimes f^{\dagger p\beta} + \right.\right.$$

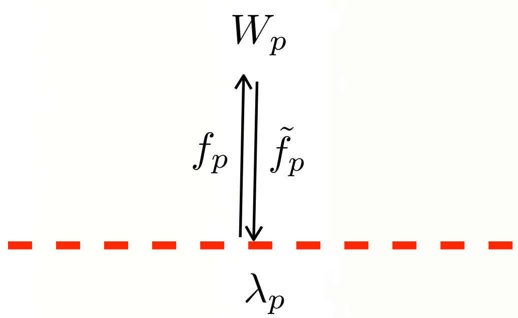
$$\left.\left.\left. + \delta(x_6) B \otimes B^\dagger + \delta(x_6 - l) B^\dagger \otimes B \right]\right)\right\}$$

$$\frac{1}{\sqrt{2}} \frac{\partial}{\partial x_6} - \mathrm{Re} H_1$$

$$\mathcal{D}^\alpha_\beta = \mathcal{D}^i (\sigma^i)^\alpha_\beta$$

# $\mathcal{D}$ -flatness conditions

$$T_0 = -\sqrt{2}\,\mathrm{Re}H_1, \quad T_1 = -\sqrt{2}\mathrm{Im}H_1, \quad T_2 + iT_3 = -\sqrt{2}H_2,$$




$$f_p = \begin{pmatrix} f_p \\ \tilde{f}_p^\dagger \end{pmatrix} = \begin{pmatrix} j_p^\dagger \\ i_p \end{pmatrix}$$

$$\begin{aligned}
\frac{dT_1}{dx_1} + [T_0, T_1] + [T_2, T_3] &= -\frac{i}{R_1}\sum_{p=1}^{k}\delta(s-\lambda_p)\left(f^p \otimes f^{\dagger p} - \tilde{f}^{\dagger p} \otimes \tilde{f}^p\right) + \\
&\quad +\delta(s)(B_{01}B_{01}^\dagger - B_{10}^\dagger B_{10}) + \delta(s-l)(B_{10}B_{10}^\dagger - B_{01}^\dagger B_{01}), \\
\frac{dT_2}{dx_1} + [T_0, T_2] + [T_3, T_1] &= -\frac{i}{R_1}\sum_{p=1}^{k}\delta(s-\lambda_p)\left(-if^p \otimes \tilde{f}^p + i\tilde{f}^{\dagger p} \otimes f^{\dagger p}\right) + \\
&\quad +\delta(s)(-iB_{01}B_{10} + iB_{10}^\dagger B_{01}^\dagger) + \delta(s-l)(iB_{10}B_{01} - iB_{01}^\dagger B_{10}^\dagger), \\
\frac{dT_3}{dx_1} + [T_0, T_3] + [T_1, T_2] &= -\frac{i}{R_1}\sum_{p=1}^{k}\delta(s-\lambda_p)\left(f^p \otimes \tilde{f}^p + \tilde{f}^{\dagger p} \otimes f^{\dagger p}\right) + \\
&\quad +\delta(s)(B_{01}B_{10} + B_{10}^\dagger B_{01}^\dagger) + \delta(s-l)(B_{10}B_{10} + B_{01}^\dagger B_{01}^\dagger)
\end{aligned}$$

Exactly the HKM of the proposed diagrams.

# Nahm Transform

Given Bow data consider Weyl Operator:

$$\mathfrak{D}: f \mapsto \begin{pmatrix} (-\frac{d}{ds} + iT_0 + \Upsilon)f \\ (J_L, I_L^\dagger)f(-\lambda) \\ (J_R, I_R^\dagger)f(\lambda) \\ (B_{01}, B_{10}^\dagger)f(l/2) \\ (-B_{10}, B_{01}^\dagger)f(-l/2) \end{pmatrix}$$

$\chi_\alpha \in E_{\lambda_\alpha}$, $v_- \in E_{-l/2}$ and $v_+ \in E_{l/2}$

cokernel of $\mathfrak{D}$ is given by $(\psi(s), \chi_L, \chi_R, v_-, v_+)$

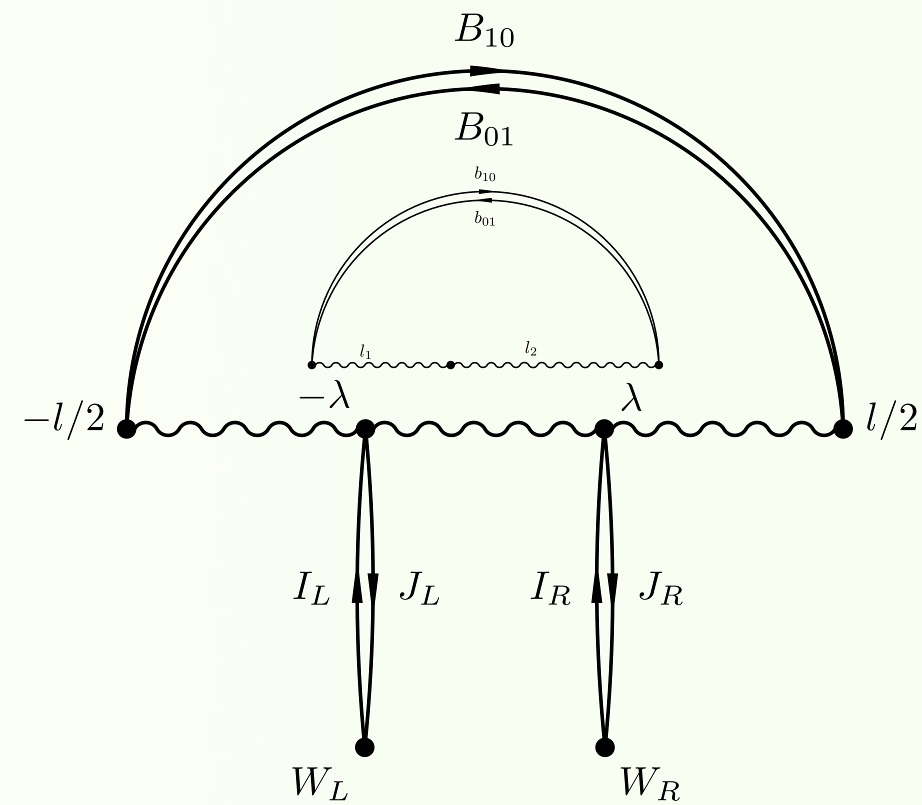
$$\boxed{\begin{aligned} &\left(\frac{d}{ds} - iT_0 + \Upsilon\right)\psi = 0, \text{ on } \mathcal{I}\backslash\{\alpha_L, \alpha_R\}, \\ &\psi(\lambda_\alpha+) - \psi(\lambda_\alpha-) = -Q_\alpha \chi_\alpha, \\ &\psi(\frac{l}{2}) \quad = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} v_-, \\ &\psi(-\frac{l}{2}) = -\begin{pmatrix} -B_{10}^\dagger \\ B_{01} \end{pmatrix} v_+. \end{aligned}}$$

$$\mathfrak{D}^\dagger = \begin{pmatrix} -D^\dagger & T^\dagger \\ T & D \end{pmatrix} \oplus \left( \bigoplus_{\alpha \in \{L,R\}} \delta_{(s-\lambda_\alpha)} \begin{pmatrix} J_\alpha^\dagger \\ I_\alpha \end{pmatrix} \right)$$
$$\oplus \left( \delta_{(s+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, \ \delta_{(s-\frac{l}{2})} \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} \right).$$

Moment map conditions are equivalent to $\mathrm{Vec}(\mathfrak{D}^\dagger \mathfrak{D}) = 0.$

# Twisted dual Weyl Operator:

given a point of the Taub-NUT space $(t_0, \vec{t}, b_{10}, b_{01})$




$$\mathfrak{D}_t^\dagger = \begin{pmatrix} -D^\dagger - t_3 & T^\dagger - t^\dagger \\ T - t & D + t_3 \end{pmatrix} \oplus \left( \bigoplus_{\alpha \in \{L,R\}} \delta(s - \lambda_\alpha) \begin{pmatrix} J_\alpha^\dagger \\ I_\alpha \end{pmatrix} \right)$$
$$\oplus \left( \delta_{(s+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger & -b_{10}^\dagger \\ -B_{01} & -b_{01} \end{pmatrix} + \delta_{(s-\frac{l}{2})} \begin{pmatrix} -b_{01}^\dagger & B_{01}^\dagger \\ b_{10} & B_{10} \end{pmatrix} \right).$$

$\psi$ a section of $E \otimes e \otimes \mathbb{C}^2 \to \mathcal{I} \backslash \{-\lambda, \lambda\}$; $v_- \in E_{-l/2} \otimes e_{l/2}$ and $v_+ \in E_{l/2} \otimes e_{-l/2}$.

For $\boldsymbol{\psi}_1 = (\psi_1(s), \chi_{L1}, \chi_{R1}^+, v_1)$, $\boldsymbol{\psi}_2 = (\psi_2(s), \chi_{L2}, \chi_{R2}, v_2)$ there is a natural Hermitian product

$$(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = v_1^\dagger v_2 + (\chi_{L1})^\dagger \chi_{L2} + (\chi_{R1})^\dagger \chi_{R2} + \int_{-l/2}^{l/2} \psi_1^\dagger(s) \psi_2(s) ds.$$

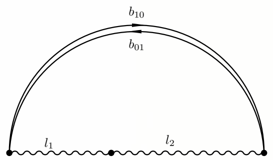
the operator $\mathbf{s}$ acting on $\boldsymbol{\psi}$ as follows

$$\boxed{\mathfrak{D}_t^\dagger \Psi = 0}$$

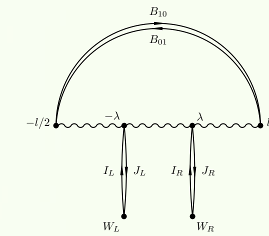
$$\mathbf{s} : (\psi(s), \chi_L, \chi_R, v) \mapsto \left( s\psi(s), -\lambda\chi_L, \lambda\chi_R, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} v \right).$$

the self-dual connection on TN is

$$A = \left( \Psi, \left( \frac{\partial}{\partial \tau} + \frac{\mathbf{s}}{V} \right) \Psi \right) d\tau + \left( \Psi, \left( \frac{\partial}{\partial x_j} + \omega_j \frac{\mathbf{s}}{V} \right) \Psi \right) dx_j$$



# Solution of $\mathfrak{D}_t^\dagger \Psi = 0$



## Taub-NUT data

$$b_- = \begin{pmatrix} -b_{01}^\dagger \\ b_{10} \end{pmatrix}, \; b_+ = \begin{pmatrix} -b_{10}^\dagger \\ -b_{01} \end{pmatrix}$$

$$b_\pm b_\pm^\dagger = |\vec{t}| \pm \cancel{t}$$

$$\mathcal{D} = F\bar{F} = (T_1 + t)^2 - z_1^2.$$

$$\mu_\pm = \sqrt{\frac{T_1 + t + \sqrt{\mathcal{D}}}{2}} \pm \sqrt{\frac{T_1 + t - \sqrt{\mathcal{D}}}{2}} \frac{\cancel{z}_1}{z_1},$$

## Instanton data is simple:

$$\vec{T}(s) = \begin{cases} \vec{T}_1 & \text{for} - l/2 < s < -\lambda \text{ or } \lambda > s > l/2 \\ \vec{T}_2 & \text{for} - \lambda < s < \lambda \end{cases}$$

$$\vec{y} = \vec{T}_2 - \vec{T}_1 = \vec{z}_1 - \vec{z}_2$$

$$B_- = \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, \; B_+ = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} \qquad Q_R = Q_+ \text{ and } Q_L = Q_-.$$

$$B_\pm B_\pm^\dagger = |\vec{T}_1| \pm \cancel{T}_1 \qquad Q_\pm Q_\pm^\dagger = y \pm \cancel{y}.$$

## Solution:

$$\Pi = \frac{1}{2g}\left(e^{-i\theta} e^{\lambda \cancel{z}_2}(y - \cancel{y}) e^{-(l/2-\lambda)\cancel{z}_1} \mu_- + e^{i\theta} e^{-\lambda \cancel{z}_2}(y + \cancel{y}) e^{(l/2-\lambda)\cancel{z}_1} \mu_+\right)$$

$$\boxed{v = \frac{1}{\sqrt{\mathcal{D}}} \begin{pmatrix} e^{i\tau/2} B_-^\dagger \mu_+ \\ e^{-i\tau/2} B_+^\dagger \mu_- \end{pmatrix} \qquad \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} Q_+^\dagger e^{-\lambda \cancel{z}_2} \\ Q_-^\dagger e^{\lambda \cancel{z}_2} \end{pmatrix} \frac{e^{-i\theta} e^{-\lambda \cancel{z}_2} e^{-(\frac{l}{2}-\lambda)\cancel{z}_1} \mu_- - e^{i\theta} e^{\lambda \cancel{z}_2} e^{(\frac{l}{2}-\lambda)\cancel{z}_1} \mu_+}{2g}.}$$

$$\boxed{\psi(s) = \begin{cases} e^{\cancel{z}_1 (s + l/2} e^{i\theta} \mu_+ & \text{for} - l/2 < s < -\lambda \\ e^{\cancel{z}_2 s} \Pi & \text{for} - \lambda < s < \lambda \\ e^{\cancel{z}_1 (s - l/2} e^{-i\theta} \mu_- & \text{for} \lambda < s < l/2 \end{cases}}$$

$$g = y \cosh 2z_2\lambda - \frac{\vec{z}_2 \cdot \vec{y}}{z_2} \sinh 2z_2\lambda$$

For this $\Psi$ $\quad (\Psi, \Psi) = m\mathbb{I}$

$$
\begin{aligned}
m =& 2 + \frac{1}{g}\Big\{ -\sqrt{\mathcal{D}}\cos 2\theta \\
&+ \frac{(T_1 + t)\sinh z_1 d - z_1 \cosh z_1 d}{z_1}\left( y\cosh 2\lambda z_2 + \frac{\vec{z}_1 \cdot \vec{z}_2 - y^2}{z_2}\sinh 2\lambda z_2 \right) \\
&+ \frac{(T_1 + t)\cosh z_1 d - z_1 \sinh z_1 d}{z_2}(z_2 \cosh 2\lambda z_2 + y\sinh 2\lambda z_2) \Big\}. \qquad (50)
\end{aligned}
$$

$$
v = \frac{1}{\sqrt{\mathcal{D}}}\begin{pmatrix} e^{i\tau/2}B_-^\dagger \mu_+ \\ e^{-i\tau/2}B_+^\dagger \mu_- \end{pmatrix}
$$

$$
\psi(s) = \begin{cases} e^{\not{z}_1(s+l/2)} e^{i\theta}\psi_L \mu_+ & \text{for } -l/2 < s < -\lambda \\ e^{\not{z}_2 s}\Pi & \text{for } -\lambda < s < \lambda \\ e^{\not{z}_1(s-l/2)} e^{i\theta}\psi_R \mu_- & \text{for } \lambda < s < l/2 \end{cases} .
$$

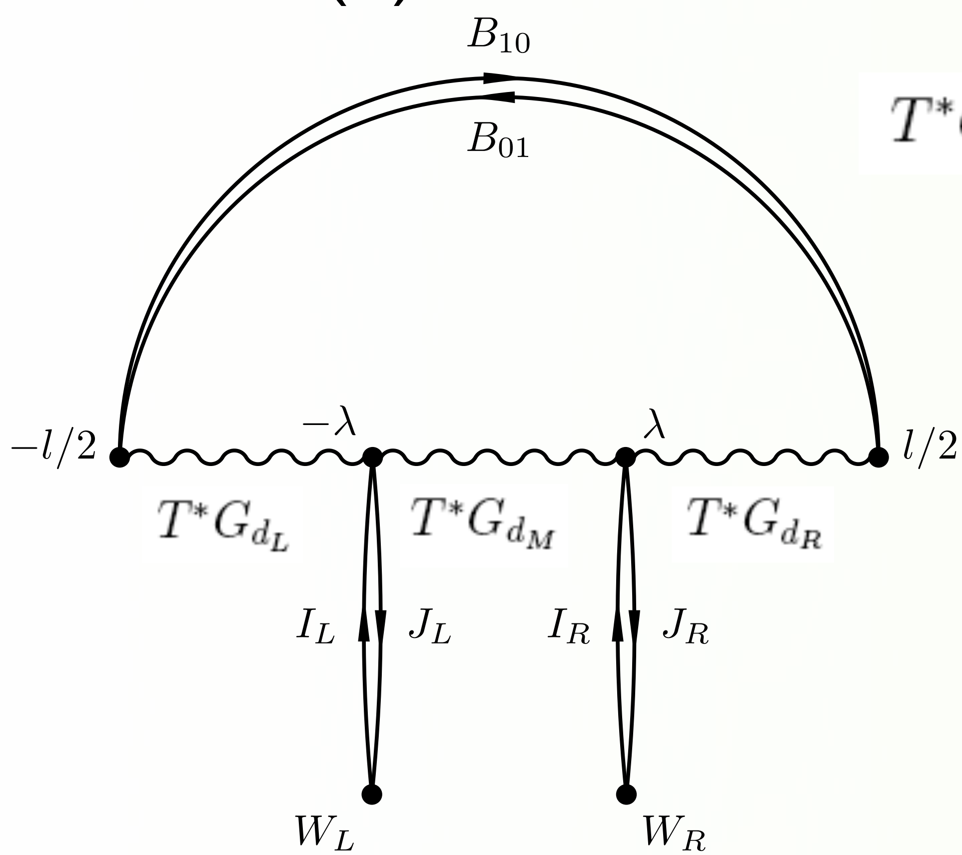
$$
\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} Q_+^\dagger e^{-\lambda \not{z}_2} \\ Q_-^\dagger e^{\lambda \not{z}_2} \end{pmatrix} \frac{e^{-i\theta} e^{-\lambda \not{z}_2} e^{-(\frac{l}{2}-\lambda)\not{z}_1}\mu_- - e^{i\theta} e^{\lambda \not{z}_2} e^{(\frac{l}{2}-\lambda)\not{z}_1}\mu_+}{2g}
$$

$$
\boxed{A = \left(\Psi, \left(\frac{\partial}{\partial\theta} + \frac{\mathbf{s}}{V}\right)\Psi\right) d\tau + \left(\Psi, \left(\frac{\partial}{\partial x_j} + \omega_j \frac{\mathbf{s}}{V}\right)\Psi\right) dx_j}
$$

# Moduli Space of N SU(2) Instantons on Taub-NUT

## One SU(2) instanton on TN

$G$ is $U(N)$




$$T^*G_{d_L} \times \mathbb{H}^N \times T^*G_{d_M} \times \mathbb{H}^N \times T^*G_{d_R} \times \mathbb{H}^{N^2} /\!/\!/ G_{-l/2} \times G_{-\lambda} \times G_{\lambda} \times G_{l/2}$$

To have algebraic description of this space introduce monodromy H on each interval:

$D_M H_M(s)=0$, $H_M(-\lambda)=1$, $H_M = H_M(\lambda)$

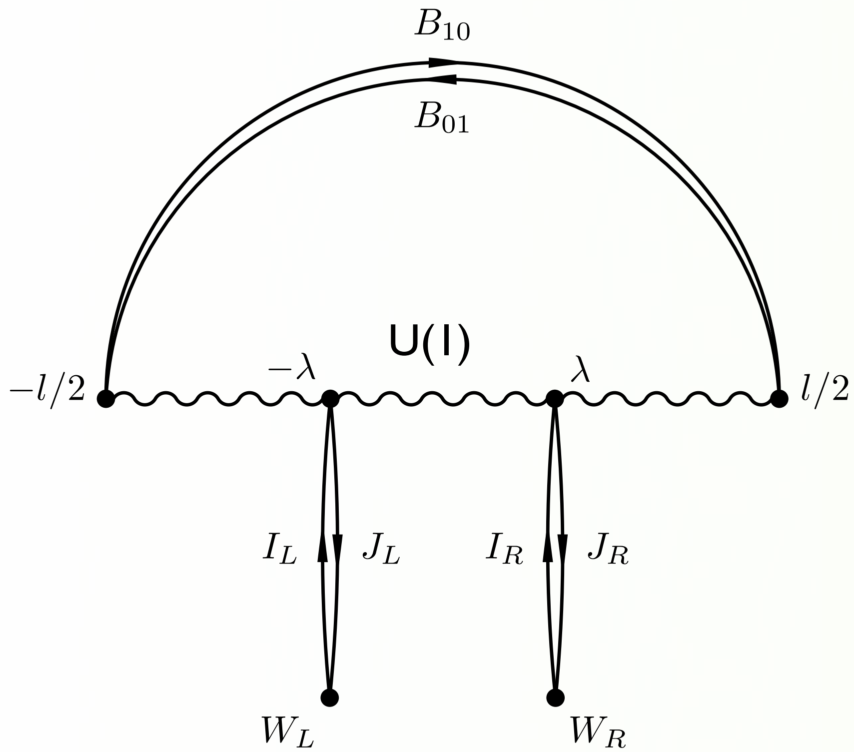
Moment maps can be written as:

$$T_R - H^{-1}_M T_M H_M = I_R J_R, \quad H^{-1}_R T_R H_R = B_{10} B_{01}$$
$$T_M - H^{-1}_L T_L H_L = I_L J_L, \quad T_L = B_{01} B_{10}$$

$\mapsto$

Up to the gauge equivalence

$$\begin{pmatrix} T_L, H_L \\ T_M, H_M \\ T_R, H_R \\ B_{01}, B_{10} \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}_{-l/2} T_L\, g_{-l/2},\ g^{-1}_{-l/2} H_L\, g_{-\lambda} \\ g^{-1}_{-\lambda} T_M\, g_{-\lambda},\ g^{-1}_{-\lambda} H_M\, g_{\lambda} \\ g^{-1}_{\lambda} T_R\, g_{\lambda},\ g^{-1}_{\lambda} H_R\, g_{l/2} \\ g^{-1}_{l/2} B_{01}\, g_{-l/2},\ g^{-1}_{-l/2} B_{10}\, g_{l/2} \end{pmatrix}$$




matches to de Boer, Hori, Ooguri, Oz
hep-th/9611063

$$ds^2 = \left(l + \frac{1}{2r_1}\right) d\vec{r}_1^{\,2} - 4\lambda d\vec{r}_1 d\vec{q} + \left(2\lambda + \frac{1}{q}\right) d\vec{q}^{\,2}$$
$$+ \frac{\left(d\theta - \frac{1}{4}\omega_r\right)^2}{l - 2\lambda + 1/q + 1/(2r)} + \frac{\left(d\alpha + \frac{1}{2}\omega_q\right)^2}{2\lambda + 1/q}$$

$$d\omega_r = *d\frac{1}{r_1}$$

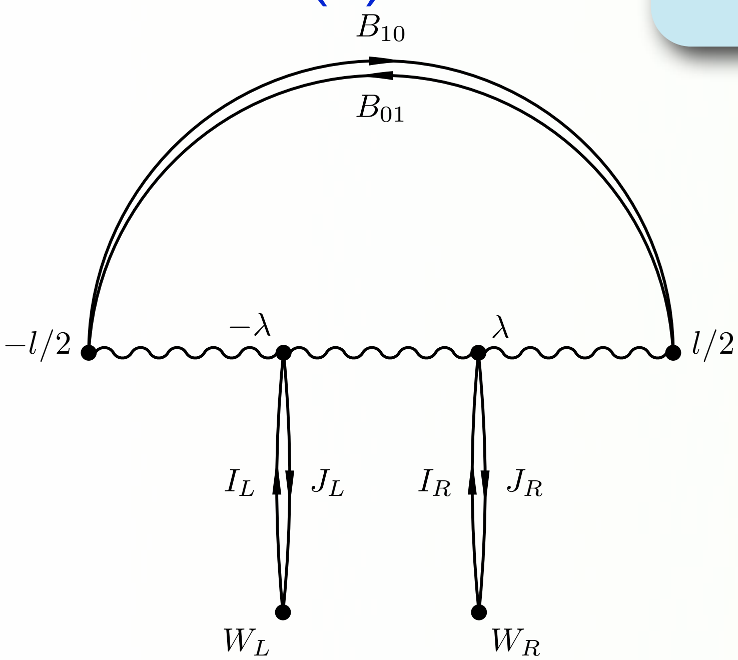
$$d\omega_q = *d\frac{1}{q}$$

$$\theta \sim \theta + 2\pi$$

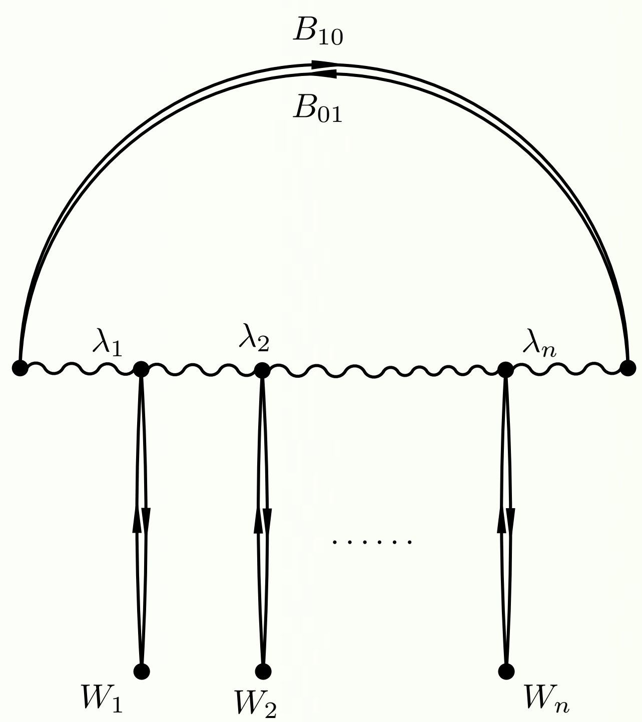
$$\alpha \sim \alpha + 2\pi$$

# Data Determining N U(m) Instantons on TN$_k$ Bow Diagrams

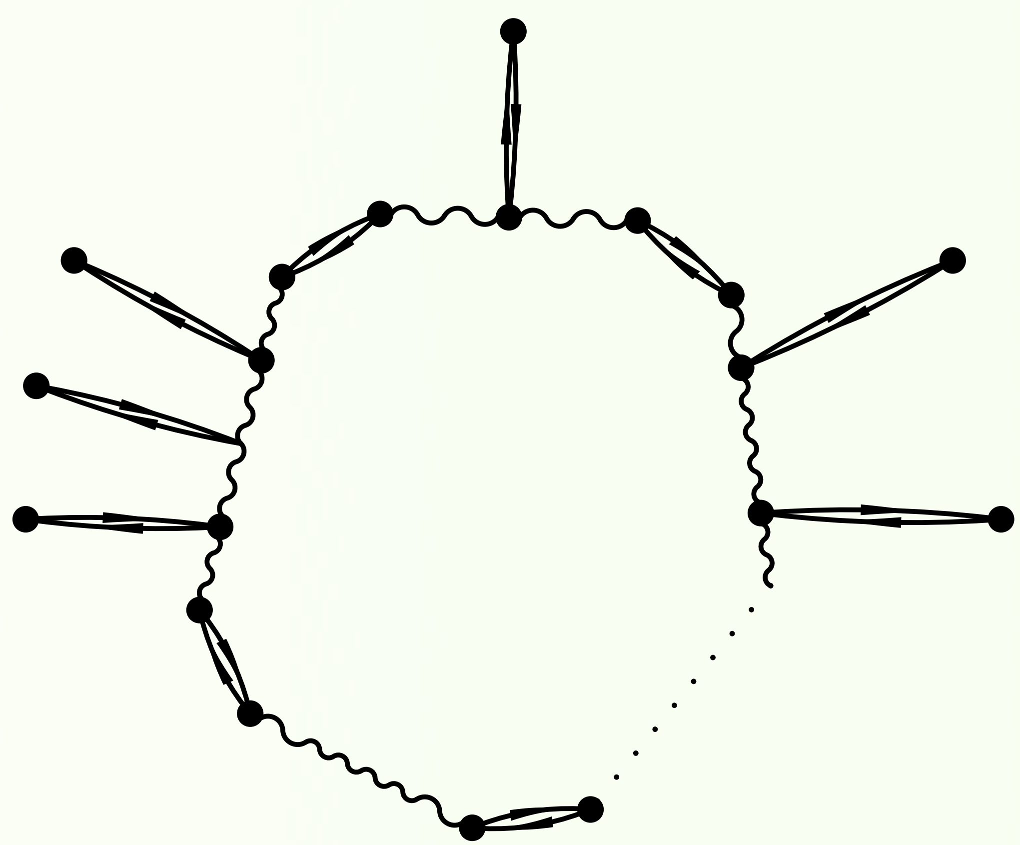
## N SU(2) on TN



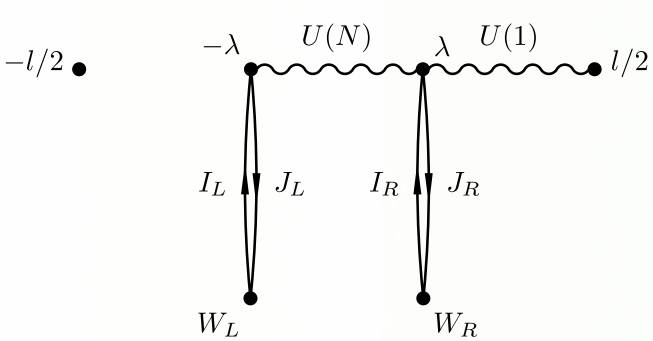

## N SU(m) on TN




## N SU(m) on multi-TN

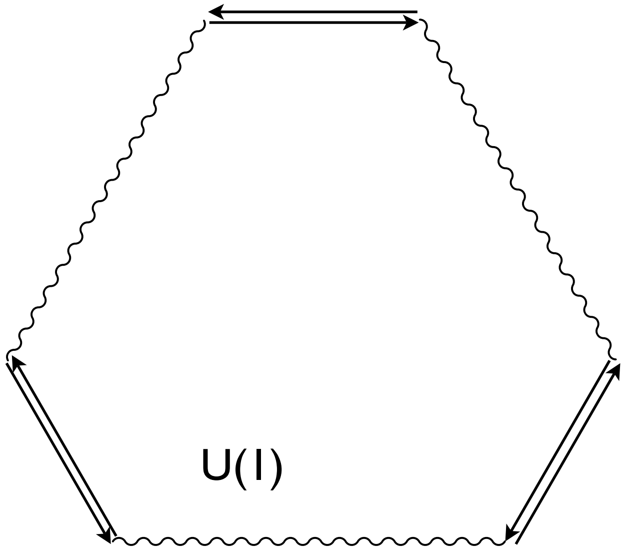


## N SU(2) “monopoles” on TN



# Instantons on ALF Spaces:

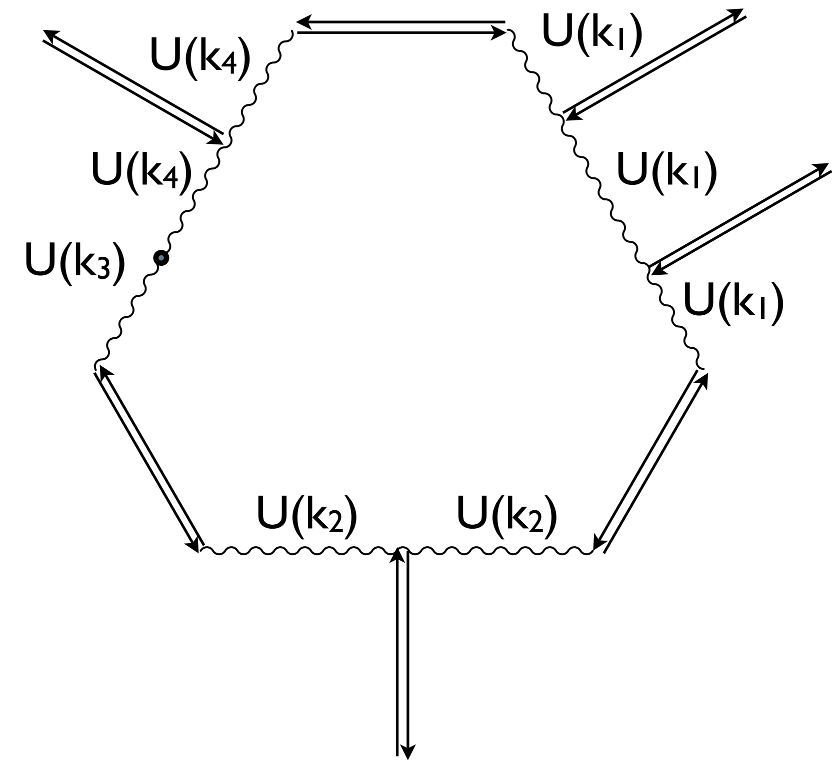
## $A_2$ ALF




U(1)

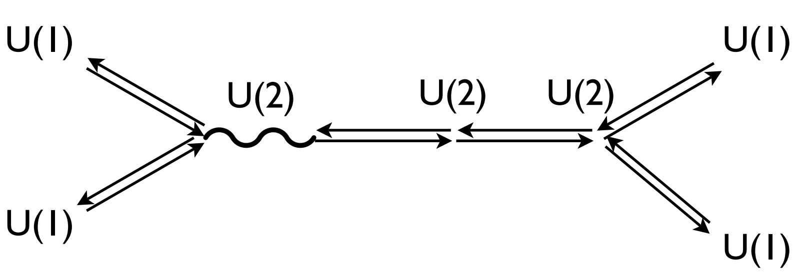
Gibbons & Rychenkova

## Instantons on $A_2$ ALF



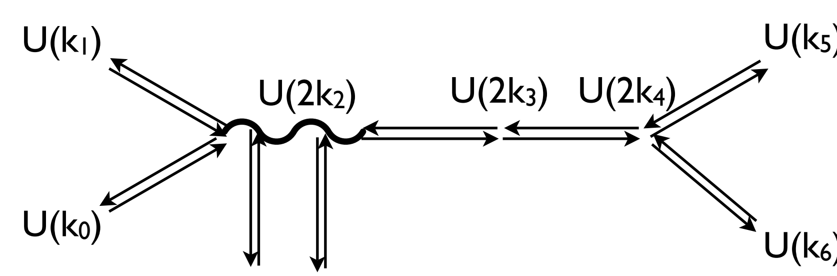

## $D_6$ ALF

U(1)

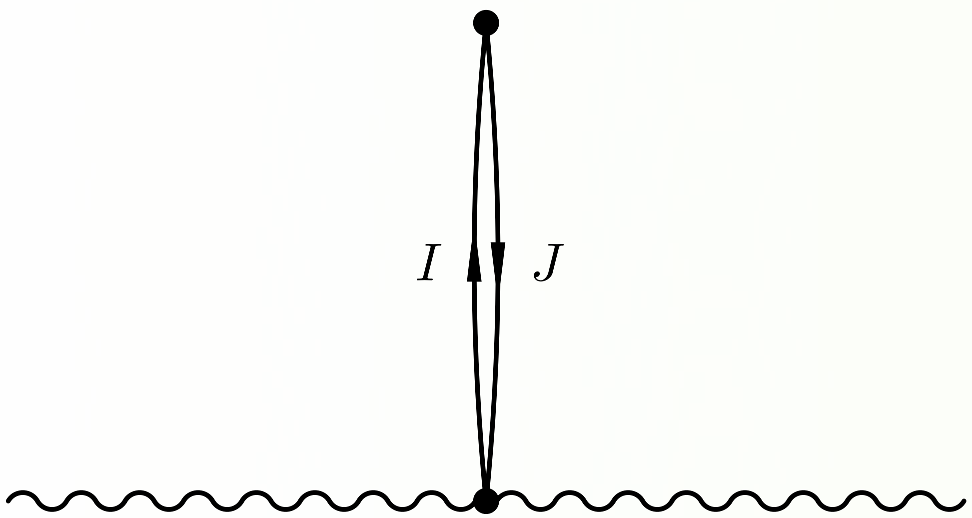



Dancer

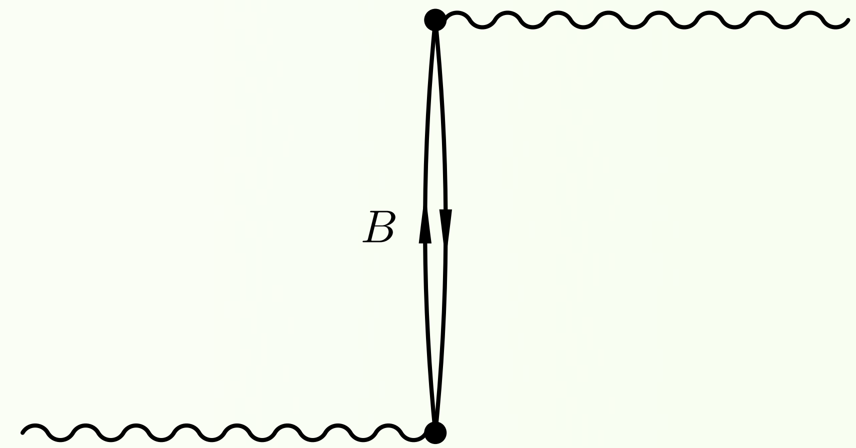
## Instantons on $D_6$ ALF



# Electric-Magnetic Duality




EM duality

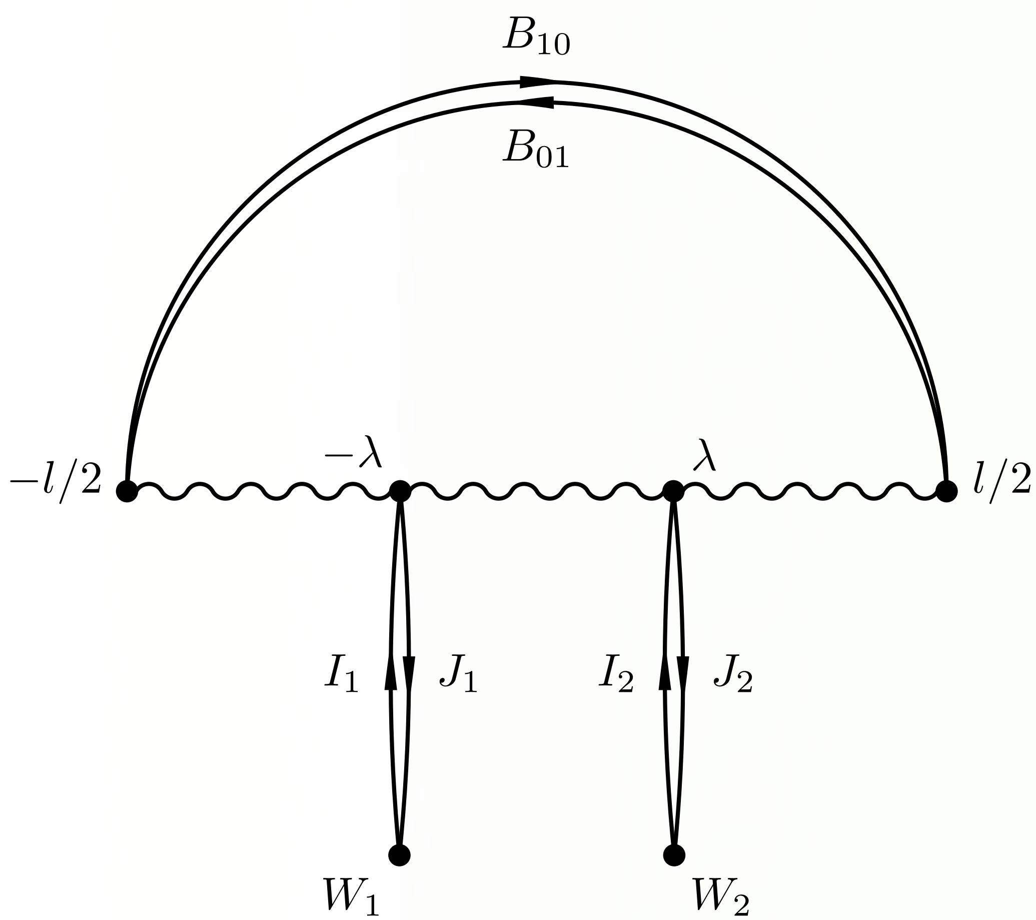



Adjoint

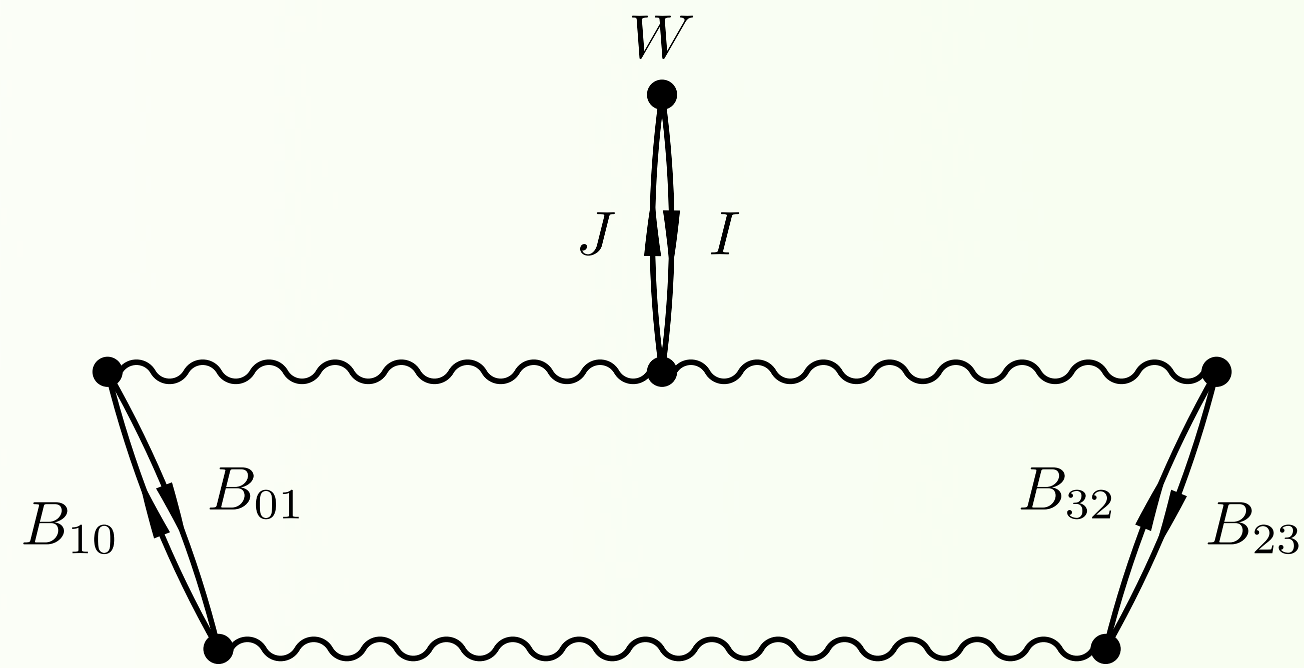
Bifundamental

# Bow Doublet

## Higgs Branch

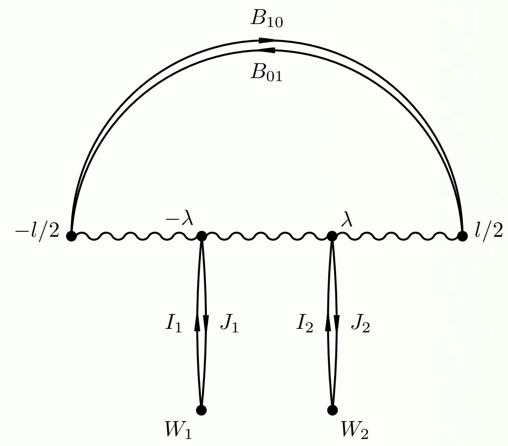



## Coulomb Branch



# Higgs Branch

$< H^2 > \neq 0, < Y^2 > = 0$



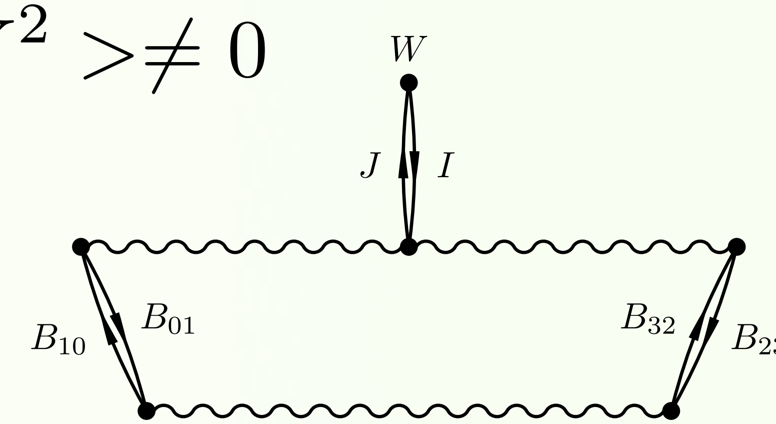

N U(m) Inst / TN$_k$

# Mixed Branch

$H_\alpha \sim I, Y_J \sim I$

# Coulomb Branch

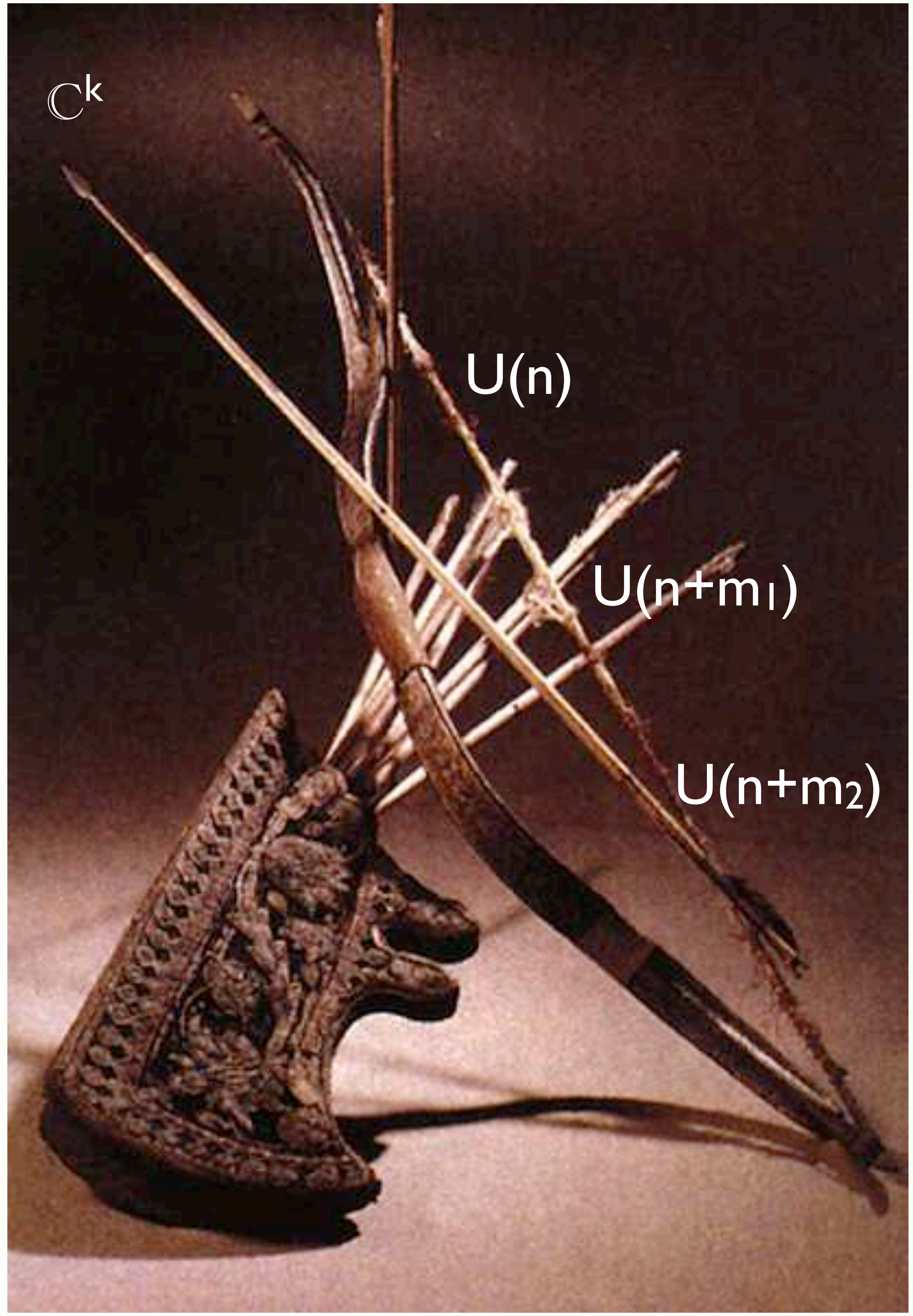
$< H^2 > = 0, < Y^2 > \neq 0$




N U(k) Inst / TN$_m$

# Summary:

1. Problem: Instantons & Monopoles
2. Explicit Monopole Solution
3. Ingredients: Arrows & Strings
4. Answer: Bow Diagrams
5. String Dualities
6. Gauge theory with Impurity walls
7. Explicit Instanton Solution
8. Moduli spaces of instantons on ALF
9. EM duality of Bows


$\mathbb{C}^k$
U(n)
U(n+m1)
U(n+m2)

$$\left(e^{-\vec{\sigma}\cdot\vec{z}_1(l-\lambda_2)}\begin{pmatrix} -b_{01}^\dagger & B_{01}^\dagger \\ b_{10} & B_{10} \end{pmatrix} + e^{\vec{\sigma}\cdot\vec{z}_2(\lambda_2-\lambda_1)}e^{\vec{\sigma}\cdot\vec{z}_1\lambda_1}\begin{pmatrix} B_{10}^\dagger & -b_{10}^\dagger \\ -B_{01} & -b_{01} \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} +$$
$$+ e^{\vec{\sigma}\cdot\vec{z}_2(\lambda_2-\lambda_1)}\begin{pmatrix} j_1^\dagger \\ i_1 \end{pmatrix}\Delta_1 + \begin{pmatrix} j_2^\dagger \\ i_2 \end{pmatrix}\Delta_2 = 0$$

# Compare to ADHM condition:

$$\begin{pmatrix} B_{10}^\dagger - b_{01}^\dagger & B_{01}^\dagger - b_{10}^\dagger \\ -B_{01} + b_{10} & B_{10} - b_{01} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} j_1^\dagger & j_2^\dagger \\ i_1 & i_2 \end{pmatrix}\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = 0$$