Nonperturbative effects in Matrix Models and Topological Strings

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collaboration with M.Mariño, R. Schiappa
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Outline

Introduction

Topological strings

Consider the A-model on a Calabi-Yau X

$$F(Q, g_s) = \sum_{d,g} N_{d,g} Q^d g_s^{2g-2}, \quad Q = \mathrm{e}^{-t}$$

- count worldsheet instantons
- perturbative in Q, g_s

$\Downarrow$ mirror symmetry $\Downarrow$

B-model on X_{mirror} $\longrightarrow$ compute $F_g(Q)$ exactly in Q
...but can we go beyond perturbation theory in g_s?

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Non-perturbative and Large Order

Why going non-perturbative?

- A better understanding of (topological) strings
  - instanton effects $\rightarrow$ dynamics?
  - new topological invariants?
- Compute *perturbative* amplitudes using non-perturbative methods?
  - WKB-like tools?

Large Order behavior & Nonperturbative effects

- QM, QFT: Standard relation between instanton effects and large-order behavior of the perturbation series
- Asymptotics of $\frac{1}{N}$-expansion of gauge theories controlled by nonperturbative corrections $\sim e^{-N}$

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- D-brane instanton effects in string dual

[Alexandrov Kazakov Kutasov]

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Applications to Topological String Theory

If the gauge theory has a string dual:
instanton effect in gauge theory $\leftrightarrow$ asymptotics of string amplitudes

- Natural non-perturbative completion
  ☞ can be tested with asymptotics of string amplitudes!
- Information about analytic structure of topological string free energy
- Nontrivial check of conjectural dualities

✍ New conjectures about asymptotics of enumerative invariants

We consider

- matrix models in double-scaling limit $\leftrightarrow$ noncritical string theory
- matrix models off criticality $\leftrightarrow$ topological strings

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Matrix Models and Topological Strings

B-model on some local CYs $\overset{\text{large N dual}}{\longleftrightarrow}$ Matrix model

[Dijkgraaf Vafa]

This also works for mirrors of toric geometries!

[Mariño; Bouchard Klemm Mariño Pasquetti]

new formalism to compute open & closed B-model amplitudes:

Topological string amplitudes F_g behave like matrix model correlators	+	Recursive, geometric reformulation of matrix model $1/N$-expansion:all information encoded in spectral curve

[Eynard Orantin]

- Spectral curve for TS on mirror of toric CY: mirror curve $\Sigma_t(u, v) = w^+w^-$
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Instanton effects and Large Order behavior

A Quantum mechanics example

Consider the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda x^4}{4}.$$

Take the perturbative expansion of the ground-state energy,

$$S_E(\lambda) = \sum_k E_k \lambda^k.$$

- S_E is in principle expected to have zero radius of convergence, $R = 0$!

  [Dyson]

  - Indeed here: $R > 0$ would imply that the perturbative series describes the physics also for $\lambda < 0$, where the state is unstable and the particle escapes $\rightarrow$ ⚡⚡

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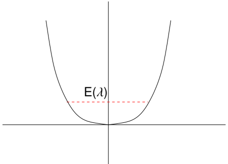

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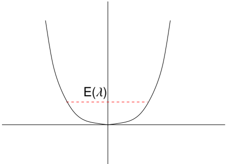

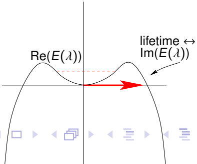

The anharmonic oscillator

[Bender Wu]

Let $S_E(\lambda) = \sum_{k\geq 0} E_k \lambda^k$ be the formal, divergent expansion of the ground state energy $E(\lambda)$.

- $E(\lambda)$ is an analytic function of the coupling λ in the cut complex plane

  [Loeffel Martin]

- $S_E(\lambda)$ is asymptotic to $E(\lambda)$

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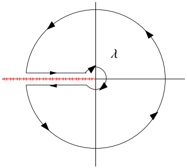
$$E(\lambda) = \frac{1}{2\pi \mathrm{i}} \oint d\lambda' \frac{E(\lambda')}{\lambda' - \lambda}$$



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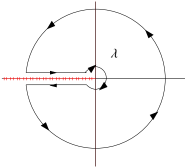
→ We can deform the Cauchy representation to the dispersion relation



$$\boxed{E_k = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{0} d\lambda' \frac{\mathrm{Disc}(E(\lambda'))}{\lambda'^{k+1}}}$$

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- This result is rigorous and exact
- The perturbation coefficients are related to the lifetime of the state in the unstable potential with negative coupling $\leftrightarrow$ instanton effect at $\lambda < 0$

$\mathrm{Disc}(E(\lambda)) = ?$

Consider $\mathcal{I}(\lambda) = \int_{-\infty}^{\infty} e^{-(x^2+\lambda x^4)} dx$:Analytic continuation to $\lambda < 0 \rightarrow$

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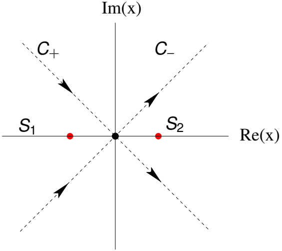

$$E_k \sim \frac{\mu_1}{2\pi} \mathcal{A}_{inst}^{-k-b} \Gamma(k+b)\left(1 + \frac{\mathcal{A}_{inst}}{(b+k-1)}\mu_2 + O\left(\frac{1}{k^2}\right)\right) \rightarrow$$

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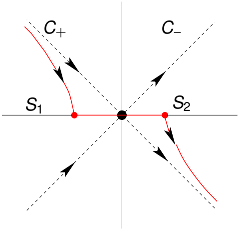

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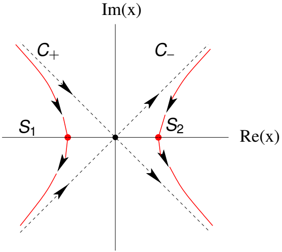

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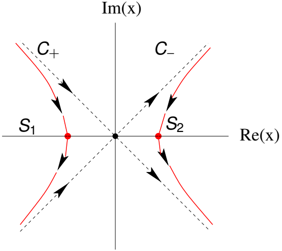

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$\uparrow$ 1-loop

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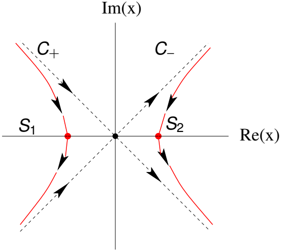

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saddle-point expansion

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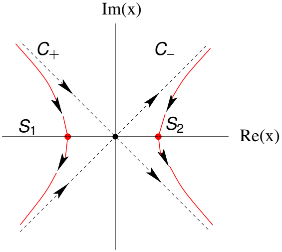

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$\mathcal{A}_{inst} = 2\int_0^{x_0} \sqrt{2V(x)}dx = -\frac{1}{3}$ $\rightarrow$ action of tunneling-instanton

$$E_k \sim \frac{\mu_1}{2\pi} \mathcal{A}_{inst}^{-k-b} \Gamma(k+b)\left(1 + \frac{\mathcal{A}_{inst}}{(b+k-1)}\mu_2 + \mathcal{O}\left(\frac{1}{k^2}\right)\right) \rightarrow$$

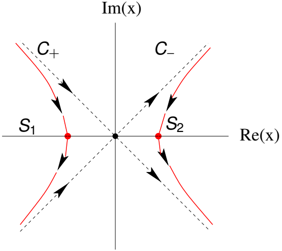
anharmonic oscillator: $E_k \sim (-1)^{k+1} \frac{\sqrt{6}}{\pi^{\frac{3}{2}}} 3^k \Gamma\left(k + \frac{1}{2}\right)$ [Bender Wu]

$$E_k = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{0} d\lambda \frac{\mathrm{Disc}(E(\lambda))}{\lambda^{k+1}}$$

- This result is rigorous and exact
- The perturbation coefficients are related to the lifetime of the state in the unstable potential with negative coupling $\leftrightarrow$ instanton effect at $\lambda < 0$

$\mathrm{Disc}(E(\lambda)) = ?$

Consider $\mathcal{I}(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-(x^2+\lambda x^4)} dx$:Analytic continuation to $\lambda < 0 \rightarrow$



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Matrix models in $1/N$ expansion

- Partition function

$$Z = \frac{1}{\mathrm{vol}(U(N))}\int dM e^{-\frac{1}{g_s}\mathrm{Tr}V(M)} = \frac{1}{N!}\int\prod_{i=1}^{N}\frac{dz_i}{2\pi}e^{N^2 V_{eff}(z_i)}$$

- effective potential $V_{eff}(z_i) = V(z_i) - 2\frac{t}{N}\sum_{i\neq j}\log|z_i - z_j| \rightarrow$ Coulomb repulsion $\rightarrow$ eigenvalues spread out over interval C
- The object we are interested in is the free energy;

$$F(t) = \sum_{g\geq 0} F_g(t) g_s^{2g-2}$$

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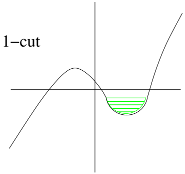

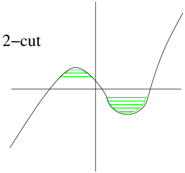

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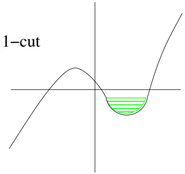

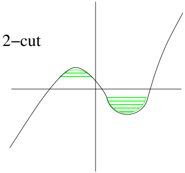

Here: Consider 1-cut case only

The planar solution

- When $N \to \infty$, the distribution of eigenvalues becomes continuous and one can write

$$V_{eff}(z) = V(z) - \frac{1}{2\pi} \int (y(z + \mathrm{i}0) - y(z - \mathrm{i}0)) \log |z - z'| dz,$$

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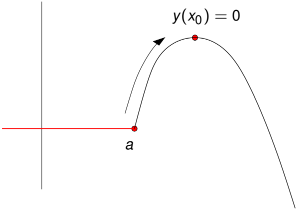
- The effective potential is constant along the cut and has a saddle point at x_0:



- Instanton configuration: an eigenvalue from the endpoint of the cut moves to the saddle of the effective potential barrier

- The instanton action is

$$\mathcal{A}_{\text{inst}} = N \int_a^{x_0} y(z) dz$$

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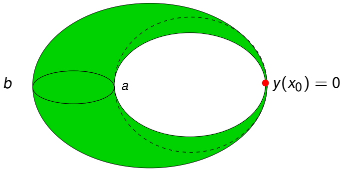
Geometrically: $\mathcal{A}_{\text{inst}}$ is a contour integral from endpoint of the cut to singularity of spectral curve



[Seiberg Shih]

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We expect a relation instantons $\leftrightarrow$ large-order analogous to the anharmonic oscillator:

$$\boxed{F_g = \frac{1}{2\pi}\int_0^\infty ds \frac{\mathrm{Disc}(F(\sqrt{s}))}{s^{g+1}} = \mu_1 \frac{\mathcal{A}_{inst}^{-b-2g}}{\pi}\Gamma(2g+b)\left(1+\frac{\mathcal{A}_{inst}}{2g+b-1}\mu_2 + O\left(\frac{1}{g^2}\right)\right)}$$

- The large-order behavior is controlled by $\mathrm{Disc}(F(g_s))$
- The discontinuity of $F(g_s)$ is again given by

$$\mathrm{Disc}(F(g_s)) = \frac{Z_N^{(1-inst)}(g_s)}{Z_N^{(0-inst)}(g_s)}$$

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1-loop, leading

2-loop, subleading

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and we find in saddle-point analysis

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- $\mathrm{Disc}(F)$ depends only on the spectral curve of the matrix model, not on the potential

  $\downarrow$ B-model formalism

  unambiguously defined for topological strings on mirrors of toric geometries

  - $a, \ b, \ x_0$ depend on 't Hooft parameter t
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String interpretation of the instanton effects

Instanton action in the double-scaling limit of matrix model $\longleftrightarrow$ disk amplitude for D-instanton in noncritical string theory $\rightarrow$ ZZ-brane

[Alexandrov Kazakov Kutasov]

$\downarrow$

difference between disk amplitudes of FZZT branes $W_{FZZT}(a) - W_{FZZT}(x_0)$

Is there a similar story for topological string theory?

$\mathcal{A}_{inst} = \int_a^{x_0} y(z)$
$= W(x_0) - W(a)$ $\longleftrightarrow$ two branes located at a, x_0 with difference between superpotentials $W(x_0) - W(a)$
$\rightarrow$ define domain wall in underlying type II theory, with tension given by $\mathcal{A}_{inst}$

☎ Unlike the B-branes, this domain wall can couple to the complex structure!

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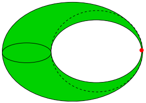

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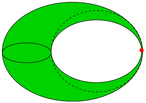

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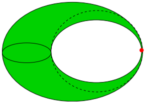

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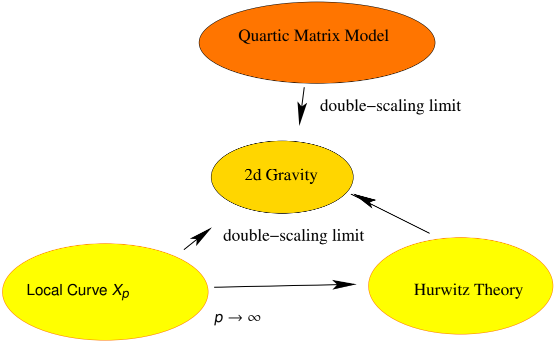

Numerical analysis:Richardson transformation

F_g are only available to limited genus, how to extract the asymptotics as $g \to \infty$? $\to$ Richardson transformation.

Given a sequence $\{S_g\}$,

$$S_g = s_0 + \frac{s_1}{g} + \frac{s_2}{g^2} + \cdots,$$

the subleading corrections up to order $\frac{1}{g^n}$ can be removed defining

$$A(g,n) = \sum_{k=0}^{N} \frac{S_{g+k}(g+k)^n(-1)^{g+n}}{k!(n-k)!}$$

If S_g truncates at $1/g^n$, this gives exactly s_0: for n=1;

$$S_g = s_0 + \frac{s_1}{g} \to A(g,1) = -(s_0 + \frac{s_1}{g}) + (s_0 + \frac{s_1}{g+1})(g+1) = s_0$$

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Numerical analysis:Richardson transformation

F_g are only available to limited genus, how to extract the asymptotics as $g \to \infty$? $\to$ Richardson transformation.
Given a sequence $\{S_g\}$,

$$S_g = s_0 + \frac{s_1}{g} + \frac{s_2}{g^2} + \cdots,$$

the subleading corrections up to order $\frac{1}{g^n}$ can be removed defining

$$A(g,n) = \sum_{k=0}^{N} \frac{S_{g+k}(g+k)^n(-1)^{g+n}}{k!(n-k)!}$$

If S_g truncates at $1/g^n$, this gives exactly s_0: for n=1;

$$S_g = s_0 + \frac{s_1}{g} \to A(g,1) = -(s_0 + \frac{s_1}{g}) + (s_0 + \frac{s_1}{g+1})(g+1) = s_0$$

- $A(g,n) = s_0 + \mathcal{O}(\frac{1}{g^{n+1}}) \to$ accelerates convergence

The quartic matrix model

Consider the matrix model with quartic potential

$$V(M) = \frac{1}{2}M^2 + \lambda M^4$$

- spectral curve:

$$y(z) = \left(1 + 8\lambda a^2 + 4\lambda z^2\right)\sqrt{z^2 - 4a^2},$$

$\pm 2a$ = endpoints of the cut,

$$a(\lambda) = \frac{1}{24\lambda}\left(-1 + \sqrt{1 + 48\lambda}\right)$$

[Brézin Itzykson Parisi Zuber]

- Critical point at $\lambda = -\frac{1}{48}$
- The free energy in $\frac{1}{N}$-expansion can be computed by standard methods

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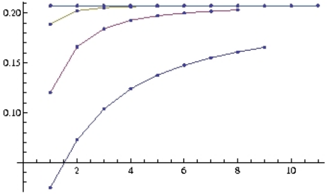
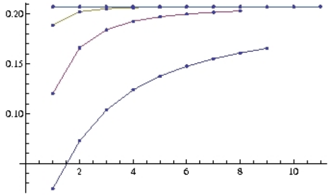


The numerical asymptotics for the instanton action, along with the matrix prediction, at $\lambda = -0.1$



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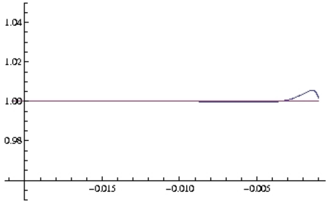
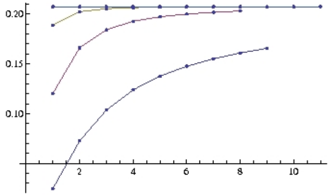


The leading asymptotics for $F_g^{\text{quart}}(\lambda)$, divided by the one-loop matrix prediction



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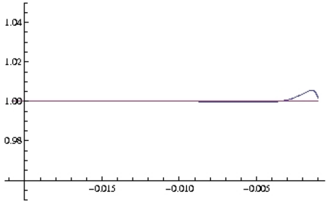

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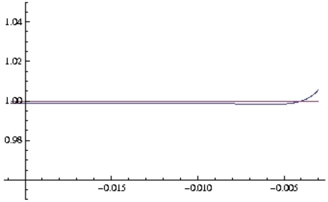

The subleading asymptotics, divided by the two-loop prediction

2d gravity

- Taking $N \to \infty$ in a standard matrix model retains only planar surfaces unless one simultaneously takes $\lambda \to \lambda_c$ where higher-genus contributions are enhanced as $F_g \propto (\lambda - \lambda_c)^{(2-\gamma)(1-g)}$: double-scaling limit→ 2d gravity

  [Gross Migdal; Douglas Shenker]

- limit discretized surface → continuum
- The perturbative amplitudes are governed by the Painlevé I equation fulfilled by the specific heat $u(z) = F''(z)$,

$$u^2 - \frac{u''}{6} = z$$

- can compute F_g to arbitrary genus
- The instanton action and 1-loop factor are

$$\mathcal{A}_{inst} = \frac{8\sqrt{3}}{5}, \quad \mu_1 = \frac{1}{8\, 3^{3/4}\sqrt{\pi}}$$

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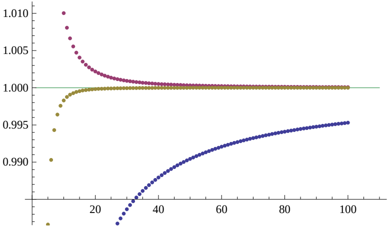

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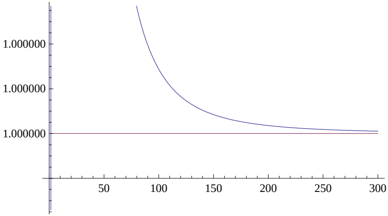

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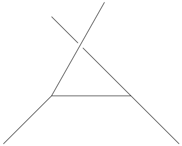
The local curve

Consider A-model topological strings on the local curve

$$X_p = \mathcal{O}(p) \oplus \mathcal{O}(2-p) \to \mathbb{P}^1, \; p \in \mathbb{Z}.$$

- This is a toric Calabi-Yau threefold with one Kähler modulus



- The potential is unstable for all $p > 2$
- The free energy can be computed using the topological vertex or local Gromov-Witten theory

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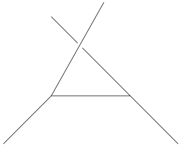

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The spectral curve corresponding to the matrix description of the mirror B-model is

$$y(z) = \frac{2}{z}\left(\left(\tanh^{-1}\left(\frac{\sqrt{(z-a)(z-b)}}{z-\frac{a+b}{2}}\right)\right) - p\tanh^{-1}\left(\frac{\sqrt{(z-a)(z-b)}}{z+\sqrt{ab}}\right)\right),$$

[Mariño]

- The endpoints of the cut a, b depend on the exponential of the Kähler parameter Q via the mirror map:

$$a = \frac{(1+\sqrt{\zeta})^2}{(1-\zeta)^p};\ b = \frac{(1-\sqrt{\zeta})^2}{(1-\zeta)^p}\ Q = (1-\zeta)^{p(p-2)}\zeta$$

- The B-model matrix formalism provides a nonperturbative completion that is testable with the large-order behaviour of the perturbative amplitudes $F_g(Q)$

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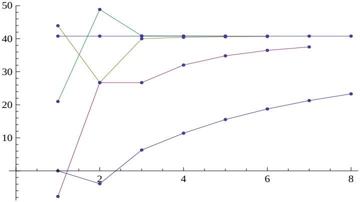

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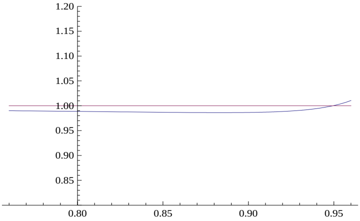
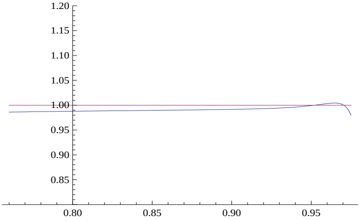


The leading asymptotics for $F_g^{p=3}$, divided by the one-loop prediction



The subleading asymptotics, divided by the two-loop prediction

Hurwitz Theory

- Hurwitz theory counts branched covers of Riemann surfaces
- obtained as a special limit of the local curve X_p:

$$p \to \infty,\ g_s \to 0,\ Q \to 0;\ g^H = Npg_s,\ Q_H = \frac{(-1)^p}{(g_s N)^2} Q$$

- $$F^H = \sum_{g\geq 0} N^{2-2g} \sum_{d\geq 0} Q_H^d H_{g,d}^{\mathbb{P}^1}(\mathbf{1}^d)^{\bullet} \frac{g_H^{2g-2+2d}}{(2g-2+2d)!}$$

- The mirror map and endpoints of the cut are given by

$$\chi e^{-\chi} = Q^H,\ a_H(\chi) = (1+\sqrt{\chi})^2,\ b_H(\chi) = (1-\sqrt{\chi})^2$$

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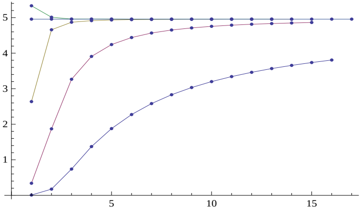

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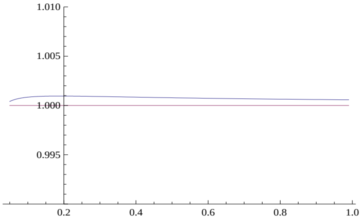

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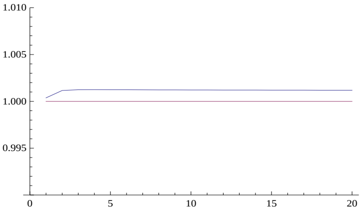

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Conclusion and Outlook

- We have computed nonperturbative effects for a generic matrix model
- The B-model formalism defines a nonperturbative completion for topological strings on local geometries
- All can be tested with the large-order behavior of the string perturbation series: agreement to very high precision

- Challenges ahead
  - multi-cut case
  - Extend B-model formalism?

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