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Nonperturbative effects in Matrix Models and Topological Strings

Marlene Weiss

CERN & ETH Zurich

collaboration with M.Mariño, R. Schiappa to appear

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Topological strings

Consider the A-model on a Calabi-Yau X

$$F(Q, g_s) = \sum_{d,g} N_{d,g} Q^d g_s^{2g-2}, \ \ Q = e^{-t}$$

- count worldsheet instantons
- perturbative in Q, g_s
 - J. mirror symmetry J.

B-model on $X_{\text{mirror}} \to \text{compute } F_g(Q)$ exactly in Q ...but can we go beyond perturbation theory in g_s ?

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Non-perturbative and Large Order

Why going non-perturbative?

- A better understanding of (topological) strings
 - Instanton effects → dynamics:
 - new topological invariants?
 - Compute perturbative amplitudes using non-perturbative methods?
 - WKB-like tools?

Large Order behavior & Nonperturbative effects

- QM, QFT: Standard relation between instanton effects and large-order behavior of the perturbation series
- Asymptotics of ¹/_N-expansion of gauge theories controlled by nonperturbative corrections ~ e^{-N}
 - 1
 - D-brane instanton effects in string dua

[Alexandrov Kazakov Kutasov]

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D-brane instanton effects in string dual

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Applications to Topological String Theory

If the gauge theory has a string dual: instanton effect in gauge theory ↔ asymptotics of string amplitudes

- Natural non-perturbative completion
 can be tested with asymptotics of string amplitudes!
- Information about analytic structure of topological string free energy
- Nontrivial check of conjectural dualities
- New conjectures about asymptotics of enumerative invariants

We consider

- matrix models off criticality ↔ topological strings

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Matrix Models and Topological Strings

B-model on some local CYs \longleftrightarrow Matrix mode

Dijkgraaf Vafa]

This also works for mirrors of toric geometries

Mariño; Bouchard Klemm Mariño Pasquetti]

new formalism to compute open & closed B-model amplitudes:

Topological string amplitudes F_g behave like matrix model correlators

Recursive, geometric reformulation of matrix model 1/*N*-expansion:all information encoded in spectral curve

[Eynard Orantin

- Spectral curve for TS on mirror of toric CY: mirror curve $\Sigma_l(u,v) = w^+w^-$
- recursive matrix model formalism → generate TS amplitudes
 - no holomorphic ambiguity
- at large radius: mirror to topological vertex, but valid anywhere in moduli space

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B-model on some local CYs $\stackrel{\text{large N dual}}{\longleftrightarrow}$ Matrix model



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Instanton effects and Large Order behavior

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda x^4}{4}$$

$$S_{E}(\lambda) = \sum_{k} E_{k} \lambda^{k}.$$

$$R = 0$$

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A Quantum mechanics example

Consider the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda x^4}{4}.$$

Take the perturbative expansion of the ground-state energy,

$$S_E(\lambda) = \sum_k E_k \lambda^k.$$

• S_E is in principle expected to have zero radius of convergence

[Dyson]

Indeed here: R > 0 would imply that the perturbative series describes the physics also for $\lambda < 0$, where the state is unstable and the particle escapes $\rightarrow 2$

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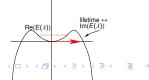
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The anharmonic oscillator

[Bender Wu]

Let $S_E(\lambda) = \sum_{k \geq 0} E_k \lambda^k$ be the formal, divergent expansion of the ground state energy $E(\lambda)$.

 $E(\lambda)$ is an analytic function of the coupling λ in the cut complex plane

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$$E(\lambda) = \frac{1}{2\pi i} \oint d\lambda' \frac{E(\lambda')}{\lambda' - \lambda}$$



The anharmonic oscillator

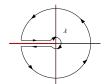
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 $E(\lambda) = \frac{1}{2\pi i} \oint d\lambda' \frac{E(\lambda')}{\lambda' - \lambda}$ \rightarrow We can deform the Cauchy representation to the dispersion relation



$$E_k = rac{1}{2\pi \mathrm{i}} \int_{-\infty}^0 d\lambda' rac{\mathrm{Disc}(E(\lambda'))}{\lambda'^{k+1}}$$

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$$E_k = \frac{1}{2\pi i} \int_{-\infty}^{0} d\lambda \frac{\operatorname{Disc}(E(\lambda))}{\lambda^{k+1}}$$

- This result is rigorous and exact
- The perturbation coefficients are related to the lifetime of the state in the unstable potential with negative coupling ↔ instanton effect at λ < 0

 $\operatorname{Disc}(E(\lambda)) = ?$

Consider $I(\lambda) = \int_{-\infty}^{\infty} e^{-(x^2 + \lambda x^2)} dx$: Analytic continuation to $\lambda < 0$

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$$\begin{array}{l} \operatorname{Disc}(E(\lambda)) = ? \\ \operatorname{Consider} I(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-(x^2 + \lambda x^4)} dx \\ \operatorname{Analytic continuation to} \ \lambda < 0 \rightarrow \\ \end{array}$$

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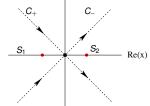
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$$E_k \sim \frac{\mu_1}{2\pi} \mathcal{R}_{inst}^{-k-b} \Gamma(k+b) (1 + \frac{\mathcal{R}_{inst}}{(b+k-1)} \mu_2 + \mathcal{O}(\frac{1}{k^2}))$$

anharmonic oscillator: $E_k \sim (-1)^{k+1} \frac{\sqrt{6}}{\pi^{\frac{3}{2}}} 3^k \Gamma(k+\frac{1}{2})$ [Bender Wil]

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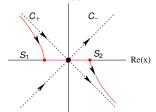
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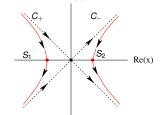
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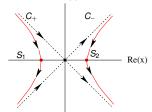
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Analogously,
$$\operatorname{Disc}(E(\lambda)) = \frac{Z^{1-inst}}{Z^{0-inst}}$$

$$=\mathrm{i}\mu_1\lambda^{-b-1}\mathrm{e}^{\frac{-\mathcal{A}_{inst}}{\lambda}}\left(1+\lambda\mu_2+O(\lambda^2)\right),$$

$$\uparrow 1\text{-loop}$$

$$E_k \sim \tfrac{\mu_1}{2\pi} \mathcal{R}_{inst}^{-k-b} \Gamma(k+b) \big(1 + \tfrac{\mathcal{R}_{inst}}{(b+k-1)} \mu_2 + O(\tfrac{1}{k^2})\big) \rightarrow$$

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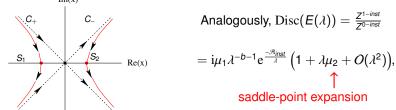
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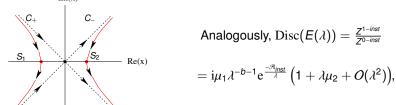
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- The perturbation coefficients are related to the lifetime of the state in the unstable potential with negative coupling \leftrightarrow instanton effect at $\lambda < 0$

Disc
$$(E(\lambda)) = ?$$

Consider $I(\lambda) = \int_{-\infty}^{\infty} e^{-(x^2 + \lambda x^4)} dx$: Analytic continuation to $\lambda < 0 \rightarrow$



$$\mathcal{A}_{inst} = 2 \int_0^{x_0} \sqrt{2V(x)} dx = -\frac{1}{3} \rightarrow \text{action of tunneling-instanton}$$

 $\left| E_k \sim \frac{\mu_1}{2\pi} \mathcal{A}_{inst}^{-k-b} \Gamma(k+b) \left(1 + \frac{\mathcal{A}_{inst}}{(b+k-1)} \mu_2 + O(\frac{1}{k^2})\right) \right| \rightarrow$

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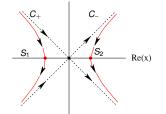
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Analogously,
$$\operatorname{Disc}(E(\lambda)) = \frac{Z^{1-\operatorname{inst}}}{Z^{0-\operatorname{inst}}}$$

$$=\mathrm{i}\mu_1\lambda^{-b-1}\mathrm{e}^{\frac{-\mathcal{H}_{inst}}{\lambda}}\left(1+\lambda\mu_2+O(\lambda^2)\right),$$

$$E_k \sim rac{\mu_1}{2\pi} \mathcal{A}_{inst}^{-k-b} \Gamma(k+b) (1 + rac{\mathcal{A}_{inst}}{(b+k-1)} \mu_2 + O(rac{1}{k^2}))
ightarrow$$

anharmonic oscillator: $E_k \sim (-1)^{k+1} \frac{\sqrt{6}}{\pi^2} 3^k \Gamma(k+\frac{1}{2})$ [Bender Wu]

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Partition function

$$Z = \frac{1}{\text{vol}(U(N))} \int dM e^{-\frac{1}{96} \text{Tr}V(M)} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dZ_i}{2\pi} e^{N^2 V_{eff}(Z_i)}$$

- effective potential $V_{eff}(z_i) = V(z_i) 2\frac{t}{N} \sum_{i \neq j} \log |z_i z_j| \rightarrow$ Coulomb repulsion \rightarrow eigenvalues spread out over interval (
- The object we are interested in is the free energy;

$$F(t) = \sum_{g \ge 0} F_g(t) g_s^{2g-2}$$

- where $t = g_s N$ is the 't Hooft parameter
- t fixed: expansion in $g_s \leftrightarrow$ expansion in $\frac{1}{N}$

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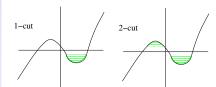
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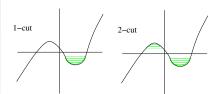
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Here: Consider 1-cut case only

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The planar solution

 When N → ∞, the distribution of eigenvalues becomes continuous and one can write

$$V_{\rm eff}(z) = V(z) - rac{1}{2\pi} \int (y(z+{
m i}0) - y(z-{
m i}0)) \log |z-z'| dz,$$

where y(z) is the spectral curve of the matrix model

[Brézin Itzykson Parisi Zuber]

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The planar solution

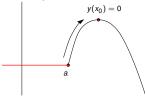
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m i}0) - y(z-{
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[Brézin Itzykson Parisi Zuber]

 The effective potential is constant along the cut and has a saddle point at x₀:



 Instanton configuration: an eigenvalue from the endpoint of the cut moves to the saddle of the effective potential barrier

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The instanton action is

$$\mathcal{A}_{\text{inst}} = N \int_{a}^{x_0} y(z) dz$$

[David; Shenker]

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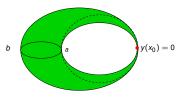
Conclusion and

The instanton action is

$$\mathcal{A}_{\text{inst}} = N \int_{a}^{x_0} y(z) dz$$

[David; Shenker]

Geometrically: \mathcal{A}_{inst} is a contour integral from endpoint of the cut to singularity of spectral curve



[Seiberg Shih]

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We expect a relation instantons ↔ large-order analogous to the anharmonic oscillator:

$$F_g = \frac{1}{2\pi} \int_0^\infty ds \frac{\operatorname{Disc}(F(\sqrt{s}))}{s^{g+1}} = \mu_1 \frac{\mathcal{A}_{\operatorname{inst}}^{-b-2g}}{\pi} \Gamma(2g+b) \Big(1 + \frac{\mathcal{A}_{\operatorname{inst}}}{2g+b-1} \mu_2 + O(\frac{1}{g^2})\Big)$$

- The large-order behavior is controlled by $\operatorname{Disc}(F(g_s))$
- The discontinuity of $F(g_s)$ is again given by

$$\operatorname{Disc}(F(g_s)) = rac{Z_N^{(1-inst)}(g_s)}{Z_N^{(0-inst)}(g_s)}$$

- = $Z_N^{(1-inst)}$ corresponds to one eigenvalue passing through the nontrivial saddle x_0 of the spectral curve \rightarrow
 - $Z_N^{(1-mst)}$ factorizes as

$$Z_N^{(1)} = Z_{N-1}^{(0)} \int_{\mathbb{C}} dz \langle \det(z - M)^2 \rangle_{N-1}^{(0)} \exp(-\frac{V(z)}{a_c})$$

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• $\langle \det(z - M)^2 \rangle$ can be expanded in terms of connected matrix correlation functions $W_{a,h}$ defined as

$$\langle \operatorname{Tr} \frac{1}{p_1 - M} \cdots \operatorname{Tr} \frac{1}{p_h - M} \rangle = \sum_{g=0}^{\infty} g_s^{2g-2+h} W_{g,h}(p_1, \dots p_h)$$

 W_{g,h} determined recursively from spectral curve by matrix model loop equations

[Ambjørn Chekhov Kristjansen Makeenko; Eynard Orantin]

The remaining ingredient is

$$\frac{Z_{N-1}^{(0)}}{Z_N^{(0)}} = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} (F_g(g_s(N-1)) - F_g(g_sN))\right)$$

and we find in saddle-point analysis

$$\operatorname{Disc}(F) = \mu_1 g_s^{1/2} \exp\left(-\frac{\mathcal{A}_{\operatorname{inst}}}{g_s}\right) \left(1 + g_s \mu_2 + \cdots\right)$$

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Disc
$$(F) = \mu_1 g_s^{1/2} \exp\left(-\frac{\mathcal{A}_{inst}}{g_s}\right) (1 + g_s \mu_2 + \cdots)$$

Explicitly:

$$\mu_1 = \frac{(a-b)}{4} \sqrt{\frac{1}{2\pi y'(x_0)((x_0-a)(x_0-b))^{\frac{3}{2}}}} e^{-\frac{1}{g_s}\mathcal{A}_{inst}}$$

 Disc(F) depends only on the spectral curve of the matrix model, not on the potential

↓ B-model formalism

- unambiguously defined for topological strings on mirrors of toric geometries
- \bullet a, b, x_0 depend on 't Hoott parameter t
- Disc(F) ~ $e^{-N\mathcal{A}_{inst}/t}$ → non-perturbative
- µ₁ has been computed before, but the result is not valid off criticality
 - [Hanada Hayakawa Ishibashi Kawai Kuroki Matsuo Tada]
- We have computed $\operatorname{Disc}(F)$ to two loops $\to \mu_1, \mu_2$

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String interpretation of the instanton effects

Instanton action in the double-scaling limit of matrix model

disk amplitude for D-instanton in noncritical string theory \rightarrow ZZ-brane

[Alexandrov Kazakov Kutasov]

Ţ

difference between disk amplitudes of FZZT branes

Wrzzz (a) – Wrzzz (x₀)

 $VV_{FZZT}(a) - VV_{FZZT}(x_0)$

Is there a similar story for topological string theory?

$$\mathcal{A}_{inst} = \int_{a}^{\infty} y(z)$$
$$= W(x_0) - W(a)$$

two branes located at a, x_0 with difference between superpotentials $W(x_0)-W(a)$

 \rightarrow define domain wall in underlying type II theory, with tension

Tunlike the B-branes, this domain wall can couple to the complex

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The quartic matrix mode 2d gravity

Hurwitz Theor

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Examples

We now test our prediction for the asymptotics in the following examples:

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The quartic matrix mode 2d gravity

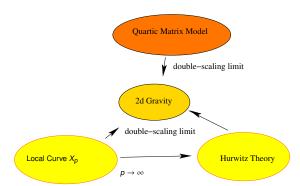
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Conclusion ar

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The quartic matrix mode

2d gravity
The local curve

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Numerical analysis:Richardson transformation

 F_g are only available to limited genus, how to extract the asymptotics as $g \to \infty$? \to Richardson transformation.

Given a sequence $\{S_g\}$

$$S_g = S_0 + \frac{S_1}{g} + \frac{S_2}{g^2} + \cdots,$$

the subleading corrections up to order $rac{1}{g^n}$ can be removed defining

$$A(g,n) = \sum_{k=0}^{N} \frac{S_{g+k}(g+k)^{n}(-1)^{g+n}}{k!(n-k)!}$$

If S_g truncates at $1/g^n$, this gives exactly s_0 : for n=1;

$$S_g = s_0 + \frac{s_1}{g} \rightarrow A(g, 1) = -(s_0 + \frac{s_1}{g}) + (s_0 + \frac{s_1}{g+1})(g+1) = s_0$$

• $A(g,n) = s_0 + O(\frac{1}{a^{n+1}}) \rightarrow \text{accelerates convergence}$

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Examples

The quartic matrix mode

The local curve

The local curve

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Numerical analysis: Richardson transformation

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• $A(q,n) = s_0 + O(\frac{1}{2^{n+1}}) \rightarrow \text{accelerates convergence}$

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Numerical analysis: Richardson transformation

 F_g are only available to limited genus, how to extract the asymptotics as $g \to \infty$? \to Richardson transformation. Given a sequence $\{S_g\}$,

$$S_g=s_0+rac{s_1}{g}+rac{s_2}{g^2}+\cdots,$$

the subleading corrections up to order $\frac{1}{g^n}$ can be removed defining

$$A(g,n) = \sum_{k=0}^{N} \frac{S_{g+k}(g+k)^{n}(-1)^{g+n}}{k!(n-k)!}$$

If S_g truncates at $1/g^n$, this gives exactly s_0 : for n=1;

$$S_g = S_0 + \frac{S_1}{g} \rightarrow A(g,1) = -(S_0 + \frac{S_1}{g}) + (S_0 + \frac{S_1}{g+1})(g+1) = S_0$$

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The quartic matrix model

Consider the matrix model with quartic potential

$$V(M) = \frac{1}{2}M^2 + \lambda M^4$$

spectral curve:

$$y(z) = (1 + 8\lambda a^2 + 4\lambda z^2)\sqrt{z^2 - 4a^2}$$

 $\pm 2a = endpoints of the cut,$

$$a(\lambda) = \frac{1}{24\lambda} \left(-1 + \sqrt{1 + 48\lambda} \right)$$

[Brezin Itzykson Parisi Zuber]

- Critical point at $\lambda = -\frac{1}{48}$
- The free energy in $\frac{1}{N}$ -expansion can be computed by standard methods

[Bessis Itzykson Zuber]

We have computed $F_{\alpha}(\lambda)$ up to genus 10:



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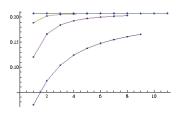
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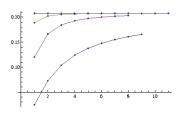
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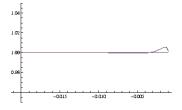
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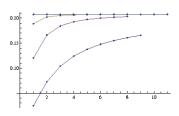
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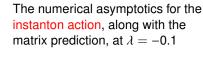
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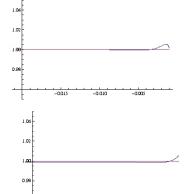
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-0.015

-0.010

-0.005

The leading asymptotics for $F_g^{\mathrm{quart}}(\lambda)$, divided by the one-loop matrix prediction

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• Taking $N \to \infty$ in a standard matrix model retains only planar surfaces unless one simultaneously takes $\lambda \to \lambda_c$ where higher-genus contributions are enhanced as $F_g \propto (\lambda - \lambda_c)^{(2-\gamma)(1-g)}$: double-scaling limit \to 2d gravity [Gross Migdal; Douglas Shenker]

- limit discretized surface → continuum
- The perturbative amplitudes are governed by the Painlevé I equation fulfilled by the specific heat u(z) = F''(z),

$$u^2 - \frac{u^{\prime\prime}}{6} = z$$

- can compute F_a to arbitrary genus
- The instanton action and 1-loop factor are



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$$\mathcal{A}_{inst} = rac{8\sqrt{3}}{5}, \ \mu_1 = rac{1}{8 \ 3^{3/4} \ \sqrt{\pi}}$$

[David]



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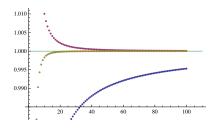
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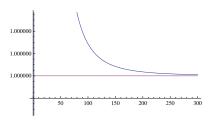
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Consider A-model topological strings on the local curve

$$X_p = O(p) \oplus O(2-p) \rightarrow \mathbb{P}^1, \ p \in \mathbb{Z}.$$

This is a toric Calabi-Yau threefold with one Kähler modulus



- The potential is unstable for all p > 2
- The free energy can be computed using the topological vertex or local Gromov-Witten theory

[Aganagic Klemm Mariño Vafa; Bryan Pandharipande]

■ double-scaling limit → 2d gravity

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The spectral curve corresponding to the matrix description of the mirror B-model is

$$y(z) = \frac{2}{z} \left(\left(\tanh^{-1} \left(\frac{\sqrt{(z-a)(z-b)}}{z - \frac{a+b}{2}} \right) \right) - p \tanh^{-1} \left(\frac{\sqrt{(z-a)(z-b)}}{z + \sqrt{ab}} \right) \right),$$
[Mariño]

 The endpoints of the cut a, b depend on the exponential of the Kähler parameter Q via the mirror map:

$$a = \frac{(1+\sqrt{\zeta})^2}{(1-\zeta)^p}; \ b = \frac{(1-\sqrt{\zeta})^2}{(1-\zeta)^p} \ Q = (1-\zeta)^{p(p-2)}\zeta$$

- The B-model matrix formalism provides a nonperturbative completion that is testable with the large-order behaviour of the perturbative amplitudes $F_a(Q)$
- Using the topological vertex, we computed F_g up to genus 9 (genus 7) for p=3 (p=4)

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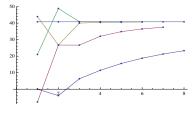
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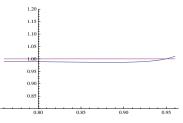
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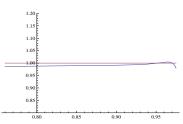
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Hurwitz Theory

- Hurwitz theory counts branched covers of Riemann surfaces
- obtained as a special limit of the local curve X_p :

$$p \to \infty, \ g_s \to 0, \ Q \to 0; \ g^H = Npg_s, \ Q_H = \frac{(-1)^p}{(g_s N)^2} Q$$

$$F^{H} = \sum_{g \geq 0} N^{2-2g} \sum_{d \geq 0} Q^{d}_{H} H^{\mathbb{P}^{1}}_{g,d} (\mathbf{1}^{d})^{\bullet} \frac{g^{2g-2+2d}_{H}}{(2g-2+2d)!}$$

The mirror map and endpoints of the cut are given by

$$\chi e^{-\chi} = Q^{rr}, \ a_H(\chi) = (1 + \sqrt{\chi})^2, \ b_H(\chi) = (1 - \sqrt{\chi})^2$$

- In the double-scaling limit (at $\chi=1$), one recovers 2d gravity
- We have computed F_g up to genus 16, finding again spectacular agreement:

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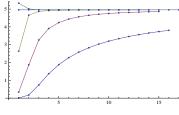
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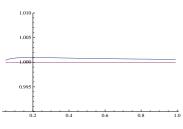
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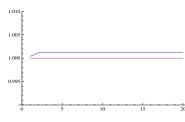
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The numerical asymptotics for the instanton action, along with the matrix prediction, at $\chi=0.5$



The leading asymptotics for $F_g^H(\chi)$, divided by the one-loop matrix prediction



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- We have computed nonperturbative effects for a generic matrix model
- The B-model formalism defines a nonperturbative completion for topological strings on local geometries
- All can be tested with the large-order behavior of the string perturbation series: agreement to very high precision

- Challenges ahead
 - multi-cut case
 - Extend B-model formalism?

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