Spectral Curve for the Heisenberg Ferromagnet and AdS/CFT

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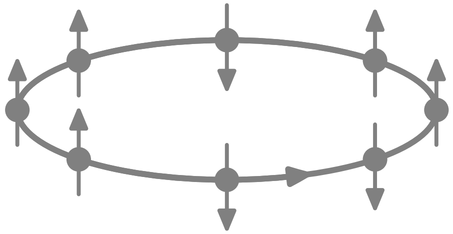

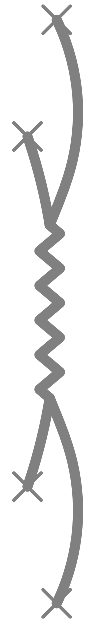

Seminar Talk
Autumn 2007

Work with Till Bargheer (to appear).
References: `hep-th/0306139,0311203,0402207,0504190,0610251.`

Introduction

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The Bethe equations are as close as one can get to a general solution.

They are a set of algebraic equations which determine the spectrum.

However, finding actual solutions to the Bethe equations is a hard problem.

Only in certain (physically useful) limits the situation may improve.

Example: the thermodynamic limit $L \to \infty$ (condensed matter).

The Heisenberg *antiferromagnet* has been studied thoroughly:

- strict limit, finite-size corrections,
- spectrum of excitations,
- correlation functions, actual measurements, . . .
- subject to fill books.

This Talk

Curiously not much was known about the Heisenberg *ferromagnet*.

This talk is about the spectrum of the Heisenberg ferromagnet.

It is useful for, at least,

- planar gauge theory,
- string theory on curved spaces,
- understanding integrable structures in the AdS/CFT correspondence.
- Someone volunteers to measure?!

Outline:

- Review of AdS/CFT integrability and strong/weak interpolation.
- Heisenberg ferromagnet and its spectral curve.
- Colliding branch cuts and stability.

AdS/CFT Integrability

AdS/CFT Correspondence

Conjectured exact duality of [Maldacena hep-th/9711200] [Gubser Klebanov Polyakov] [Witten hep-th/9802150]

- IIB superstring theory on $AdS_5 \times S^5$ and
- Four-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory (CFT).

Symmetry groups match: $\widetilde{\mathrm{PSU}}(2,2|4)$.

Holography: Boundary of AdS_5 is conformal $\mathbb{R}^4$.

Prospects:

- More general AdS/CFT may explain aspects of QCD strings.
- Study aspects of quantum gravity with QFT means.

No proof yet!

Many qualitative comparisons. Quantitative tests missing.

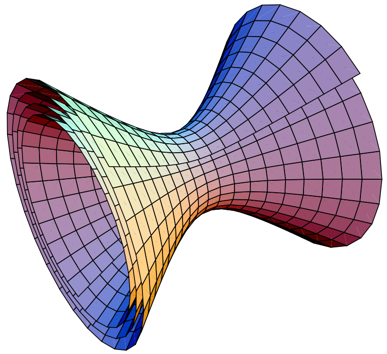
Would like to verify quantitatively.

One prediction: **Matching of spectra.**

Central motivation for this talk. Goal: Obtain spectra on both sides.

Spectrum of AdS/CFT

String Theory: $AdS_5 \times S^5$ background



States: Solutions X of classical equations of motion plus quantum corrections.

Energy: Charge E_X for translation along AdS-time.

Gauge Theory: Conformal $\mathcal{N} = 4$ SYM

States: Local operators. Local, gauge-inv. combinations of the fields, e.g.

$$\mathcal{O} = \operatorname{Tr} \Phi_1 \Phi_2 (\mathcal{D}_1 \mathcal{D}_2 \Phi_2)(\mathcal{D}_1 \mathcal{F}_{24}) + \ldots .$$

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle = C|x-y|^{-2D_{\mathcal{O}}(\lambda)}.$$

AdS/CFT: String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$?!

Strong/Weak Duality

Problem: Models have coupling constant: λ. Strong/weak duality.

- Perturbative gauge theory at $\lambda \to 0$.

$$D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \ldots .$$

D_ℓ: Contribution at ℓ (gauge) loops. Limit: 3 or 4 (or 5?!) loops.

- Perturbative regime of strings at $\lambda \to \infty$.

$$E(\lambda) = \sqrt{\lambda}\, E_0 + E_1 + E_2/\sqrt{\lambda} + \ldots .$$

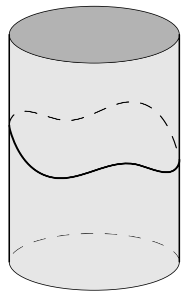
E_ℓ: Contribution at ℓ (world-sheet) loops. Limit: 1 or 2 loops.

Tests impossible unless quantities are known at finite λ.
Cannot compare, not even approximately.

Planar Limit

- Simplifications & surprises [’t Hooft Nucl. Phys. B72, 461] [Lipatov hep-th/9311037] [Faddeev Korchemsky] [Anastasiou, Bern Dixon, Kosower]
- AdS/CFT integrability [Lipatov ICTP 1997] [Mandal Suryanarayana Wadia] [Minahan Zarembo] [NB Kristjansen Staudacher] [Bena Polchinski Roiban] [NB, Staudacher hep-th/0307042] . . .

String Theory: $g_\mathrm{s} = 0$.

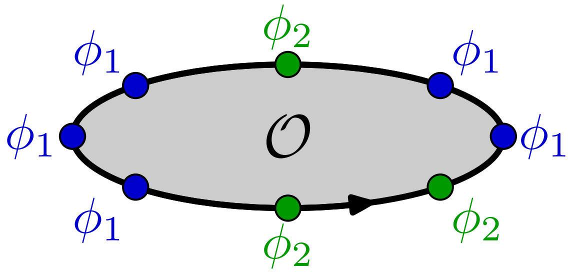


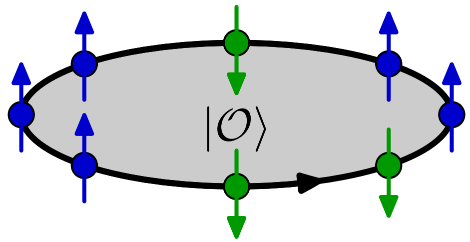
- Strictly cylindrical worldsheet.
- No string splitting or joining.

Gauge Theory: $N_\mathrm{c} = \infty$. Only single-trace operators relevant.

- Translate single-trace operators to spin chain states, e.g.

$$\mathcal{O} = \mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2 \qquad |\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$





- Energy spectrum: Eigenvalues of spin chain Hamiltonian. Integrable!!

Bethe Equations

Bethe equations to determine spectrum. [NB, Staudacher hep-th/0504190] [NB hep-th/0511082] [NB, Eden Staudacher]

coupling constant

$$g^2 = \frac{\lambda}{16\pi^2}$$

relations between u and $x^\pm$

$$u = x^+ + \frac{1}{x^+} - \frac{i}{2g} = x^- + \frac{1}{x^-} + \frac{i}{2g}$$

magnon momentum and energy

$$e^{ip} = \frac{x^+}{x^-}, \qquad e = -igx^+ + igx^- - \tfrac{1}{2}$$

total energy

$$E = L + \tfrac{1}{2}N + \tfrac{1}{2}\dot{N} + 2\sum_{j=1}^{K}\left(\frac{ig}{x_j^+} - \frac{ig}{x_j^-}\right)$$

local charges

$$q_r(x^\pm) = \frac{1}{r-1}\left(\frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}}\right), \qquad Q_r = \sum_{j=1}^{K} q_r(x_j^\pm)$$

scattering factor σ with coefficients $c_{r,s}$

$$\sigma_{12} = \exp\left(i\sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} c_{r,s}(g)\,\big(q_r(x_1^\pm)\,q_s(x_2^\pm) - q_r(x_2^\pm)\,q_s(x_1^\pm)\big)\right)$$

Bethe equations

$$1 = \prod_{j=1}^{K}\frac{x_j^+}{x_j^-}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{\dot{M}}\frac{\dot{w}_k - \dot{w}_j - ig^{-1}}{\dot{w}_k - \dot{w}_j + ig^{-1}}\prod_{j=1}^{\dot{N}}\frac{\dot{w}_k - \dot{y}_j - 1/\dot{y}_j + \frac{i}{2}g^{-1}}{\dot{w}_k - \dot{y}_j - 1/\dot{y}_j - \frac{i}{2}g^{-1}}$$

$$1 = \prod_{j=1}^{\dot{M}}\frac{\dot{y}_k + 1/\dot{y}_k - \dot{w}_j + \frac{i}{2}g^{-1}}{\dot{y}_k + 1/\dot{y}_k - \dot{w}_j - \frac{i}{2}g^{-1}}\prod_{j=1}^{K}\frac{\dot{y}_k - x_j^+}{\dot{y}_k - x_j^-}$$

$$1 = \left(\frac{x_k^-}{x_k^+}\right)^L\prod_{\substack{j=1\\ j\neq k}}^{K}\left(\frac{u_k - u_j + ig^{-1}}{u_k - u_j - ig^{-1}}\,\sigma_{12}^2\right)\prod_{j=1}^{\dot{N}}\frac{x_k^- - \dot{y}_j}{x_k^+ - \dot{y}_j}\prod_{j=1}^{N}\frac{x_k^- - y_j}{x_k^+ - y_j}$$

$$1 = \prod_{j=1}^{M}\frac{y_k + 1/y_k - w_j + \frac{i}{2}g^{-1}}{y_k + 1/y_k - w_j - \frac{i}{2}g^{-1}}\prod_{j=1}^{K}\frac{y_k - x_j^+}{y_k - x_j^-}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{w_k - w_j - ig^{-1}}{w_k - w_j + ig^{-1}}\prod_{j=1}^{N}\frac{w_k - y_j - 1/y_j + \frac{i}{2}g^{-1}}{w_k - y_j - 1/y_j - \frac{i}{2}g^{-1}}$$

magic coefficients with $c_{r,s} = \mathcal{O}(g^3)$, $c_{r,s} = g\delta_{r+1,s} + \mathcal{O}(1/g^0)$

$$c_{r,s}(g) = 2\sin[\tfrac{1}{2}\pi(s-r)]\,(r-1)(s-1)\int_0^\infty \frac{dt}{t}\,\frac{\mathrm{J}_{r-1}(2gt)\,\mathrm{J}_{s-1}(2gt)}{e^t - 1}$$

Agrees with perturbative gauge and string theory. [NB, Staudacher hep-th/0504190] [Arutyunov Frolov Staudacher]

Asymptotic $\mathcal{O}(e^{-*L})$ spectrum only. [NB, Dippel Staudacher] [Schäfer-Nameki Zamaklar, Zarembo] [Kotikov, Lipatov, Rej Staudacher, Velizhanin]

Universal Dimension

Use Bethe equations to compute some anomalous dimension.

Useful object: Twist-2 operators with spin S

$$\mathcal{O}^S \simeq \sum_{n=0}^{S} c_{S,n}\,\mathrm{Tr}(\mathcal{D}^n\mathcal{X})\,(\mathcal{D}^{S-n}\mathcal{Y}).$$

Logarithmic scaling at large spin $D \sim D_{\mathrm{uni}} \log S$ (QCD) [Sterman NPB281,310][Moch Vermaseren Vogt]

$$D^{(1)}_{\mathrm{uni}} = 8C_{\mathrm{A}}, \quad D^{(2)}_{\mathrm{uni}} = (\tfrac{536}{9} - \tfrac{8}{3}\pi^2)C_{\mathrm{A}}^2 - \tfrac{10}{9}C_{\mathrm{A}}n_{\mathrm{f}}, \quad D^{(3)}_{\mathrm{uni}} = \ldots.$$

Call coefficient D_{uni} the universal (cusp/soft) anomalous dimension.

Prediction for universal anomalous dimension from Bethe equations [NB, Eden Staudacher]

$$\pi^2 D_{\mathrm{uni}}(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{96} + \frac{11\lambda^3}{23040} - \Big(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\Big)\lambda^4 \pm \ldots.$$

Universal Dimension from Bethe Equations

Compute universal dimension using Bethe equations. Integral eq.: [Eden Staudacher]

$$\psi(x) = K(x,0) - \int_0^\infty K(x,y)\,\frac{dy\,y}{e^{y/2g}-1}\,\psi(y).$$

Kernel $K = K_0 + K_1 + K_{\mathrm{d}}$ with [NB, Eden Staudacher]

$$K_0(x,y) = \frac{x\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - y\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_1(x,y) = \frac{y\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - x\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_{\mathrm{d}}(x,y) = 2\int_0^\infty K_1(x,z)\,\frac{dz\,z}{e^{z/2g}-1}\,K_0(z,y)\,.$$

Universal anomalous dimension: $D_{\mathrm{uni}} = 16g^2\psi(0)$.

Gluon Scattering Amplitudes

Use gluon scattering to verify Bethe equations at weak coupling.

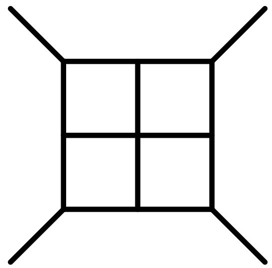
Four-gluon scattering amplitude obeys "iteration" relation [Anastasiou, Bern Dixon, Kosower][Bern Dixon Smirnov]

$$A(p,\lambda) \simeq A^{(0)}(p) \exp\left(2D_{\text{uni}}(\lambda) M^{(1)}(p)\right).$$

Same relation from string theory. [Alday Maldacena]. . .

Cusp dimension D_{uni} evaluated using unitarity methods [Bern, Czakon, Dixon Kosower, Smirnov][Cachazo Spradlin Volovich]

- four dimensions,
- four supersymmetries,
- four legs,
- four loops.

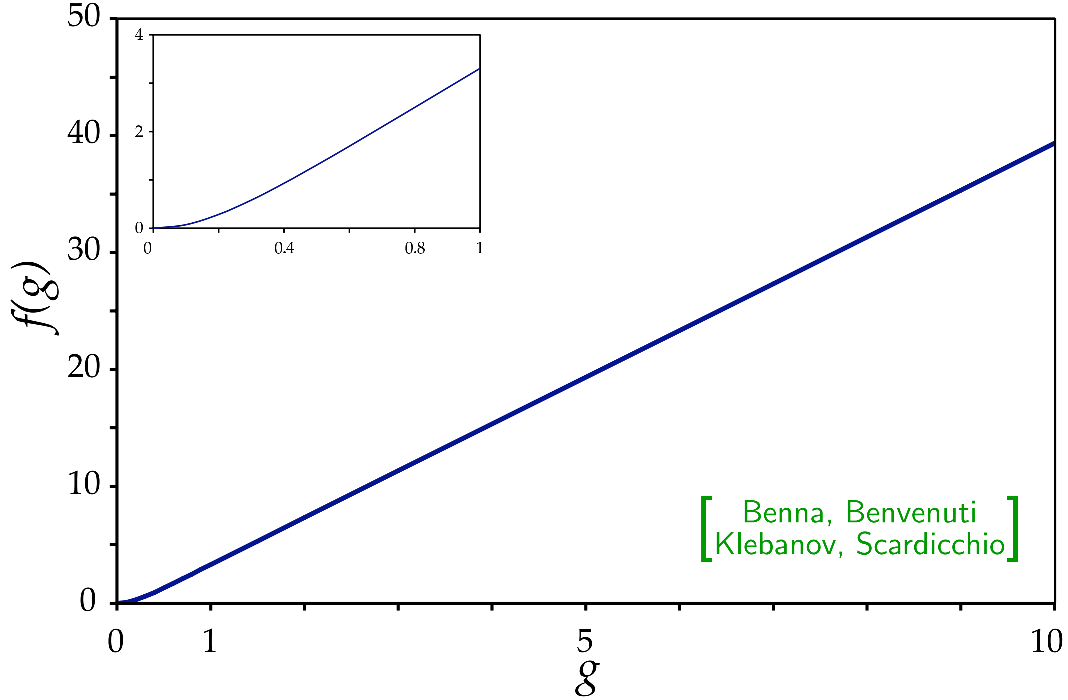


$$\pi^2 D_{\text{uni}}(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{96} + \frac{11\lambda^3}{23040} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\lambda^4 \pm \ldots .$$

Agrees with 4-loop prediction from Bethe equations.

Strong Coupling

Numerical evaluation of cusp dimension at finite coupling. [Benna, Benvenuti Klebanov, Scardicchio]



- Numerical extrapolation at large λ. [Benna, Benvenuti Klebanov, Scardicchio]
- Analytical LO and NLO. [Kotikov Lipatov] [Alday, Arutyunov Benna, Eden, Klebanov] [Kostov Serban Volin] [Beccaria De Angelis Forini] [Casteill Kristjansen]
- Asymptotic expansion at arbitrary orders [Basso Korchemsky Kotański]

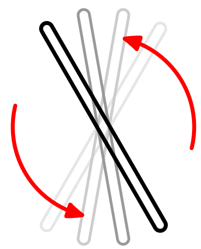
$$\pi E_{\text{uni}}(\lambda) = \sqrt{\lambda} - 3\log 2 - \beta(2)/\sqrt{\lambda} + \ldots .$$

Spinning Strings

Universal dimension from spinning string solutions. [Gubser Klebanov Polyakov]

Classical folded string rotating in $AdS_3 \subset \mathbb{R}^{2,2}$

$$\vec{Y}(\sigma,\tau) = \begin{pmatrix} \cosh\rho(\sigma)\cos(\epsilon\tau) \\ \cosh\rho(\sigma)\sin(\epsilon\tau) \\ \sinh\rho(\sigma)\cos(\omega\tau) \\ \sinh\rho(\sigma)\sin(\omega\tau) \end{pmatrix}.$$



Solve EOM and Virasoro constraint: Energy E as a function of spin S.

First quantum correction: Sum over fluctuation modes. [Frolov Tseytlin]

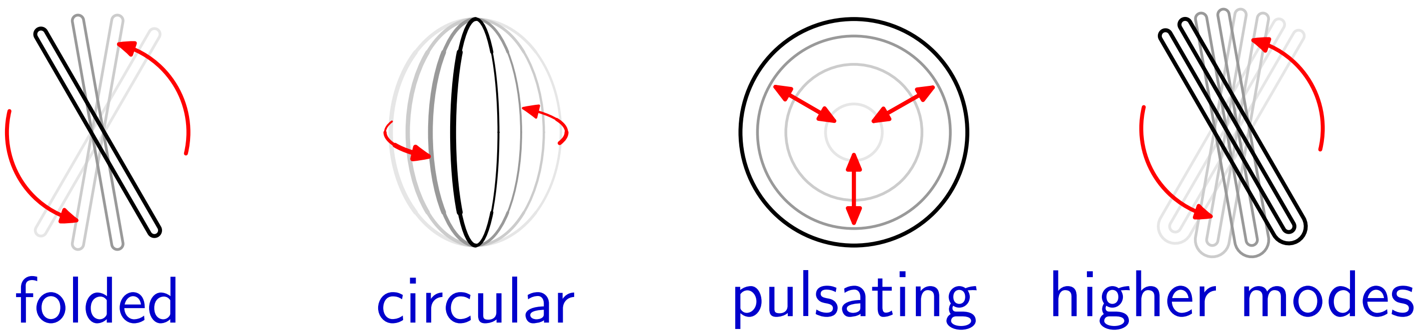
Universal energy $E = S + E_{\text{uni}} \log S + \ldots$ from NNLO string theory [Roiban Tirziu Tseytlin]

$$\pi E_{\text{uni}}(\lambda) = \sqrt{\lambda} - 3\log 2 - (\beta(2) + \infty)/\sqrt{\lambda} + \ldots .$$

Agrees with Bethe equations.

Towards Spectral Curves

Many spinning strings have been found: [Gubser Klebanov Polyakov] [Frolov Tseytlin] [Minahan hep-th/0209047] [Frolov Tseytlin] ...



- Charges are algebraic & elliptic functions. Genus 0 & 1 surfaces?!
- All of them should arise from (a limit of) Bethe equations.

Spectral Curves for AdS/CFT [Kazakov, Marshakov Minahan, Zarembo] [NB, Kazakov Sakai, Zarembo] [NB, Kazakov Sakai, Zarembo] [NB, Staudacher hep-th/0504190]

- Ferromagnetic thermodynamic limit of Bethe eq. $\rightarrow$ spectral curves.
- Classical string theory $\rightarrow$ spectral curves.
- Genus of 2D spectral curve determines class of functions.

Spectral curve $\rightarrow$ solution of Bethe equations/string E.O.M.: **Stability?**

Heisenberg Ferromagnet

Heisenberg Spin Chain

Choose a simpler spin chain model to study spectral curves & stability:
The original Heisenberg™ spin chain. [Heisenberg Z. Phys. A49, 619]
Sector of one-loop planar $\mathcal{N} = 4$ gauge theory. [Minahan Zarembo]

Space of states: Tensor product of L spin-$\frac{1}{2}$ modules of $\mathfrak{su}(2)$. Example:

$$|\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle \in (\mathbb{C}^2)^{\otimes L}$$

Periodic nearest-neighbour Hamiltonian: $\mathcal{H} : (\mathbb{C}^2)^{\otimes L} \to (\mathbb{C}^2)^{\otimes L}$

$$\mathcal{H} = \sum_{k=1}^{L} \tfrac{1}{2}(1 - \sigma_k \cdot \sigma_{k+1}) = \sum_{k=1}^{L} (1 - \mathcal{P}_{k,k+1}).$$

Spectrum of eigenvalues bounded between [Hulthén 1938]

- ferromagnetic state: $|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$: energy $E = 0$.
- antiferromagnetic state: "$|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$": energy $E \approx L \log 4$.

Bethe Equations

Bethe equations describe exact spectrum of the Heisenberg chain. [Bethe Z. Phys. 71, 205]

Consider a set of K distinct complex numbers: $\{u_k\}$, $k = 1, \ldots, K$.

The u_k represent spin flips (magnons) above the ferromagnetic vacuum.

A magnon rapidity along the chain is given by $u = \frac{1}{2}\cot(\frac{1}{2}p)$.

Bethe equations (periodicity of multi-magnon wave function)

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}.$$

For each solution there is exactly one ($\mathfrak{su}(2)$ highest-weight) eigenstate.

Its overall spin chain momentum & energy eigenvalue is given by

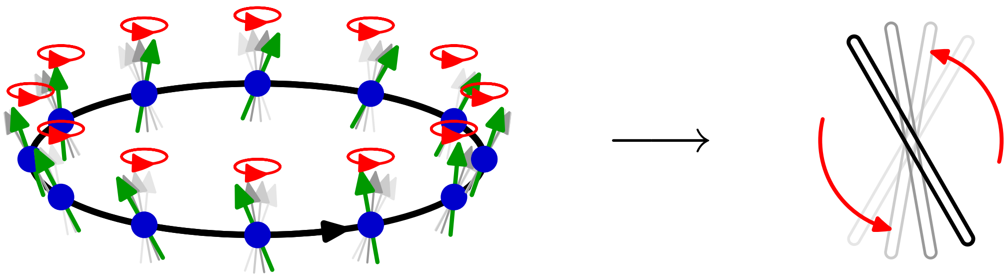
$$\exp(iP) = \prod_{k=1}^{K} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, \qquad E = \sum_{j=1}^{K} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right).$$

Ferromagnetic Thermodynamic Limit

The ferromagnetic thermodynamic limit: [Sutherland Phys. Rev. Lett. 74, 816 (1995)] [NB, Minahan Staudacher Zarembo]

- Number of spin sites $L \to \infty$.
- Number of spin flips $K \to \infty$ above ferromagnetic vacuum.
- Ratio $\alpha = K/L$ fixed.
- Only IR modes: coherent spins, energy $E = \tilde{E}/L \sim 1/L$.

Coherent spins for states $|\uparrow\rangle, |\downarrow\rangle$ specified by points on S^2 [Kruczenski hep-th/0311203]



Effective model for Heisenberg chain in the thermodynamic limit:

⋆ Classical Landau-Lifshitz sigma model on S^2. [Kruczenski hep-th/0311203]

Landau-Lifshitz Model

2D non-relativistic sigma model on S^2. Fields θ, ϕ. Lagrange function

$$L[\theta,\phi,\dot{\phi}] = -\frac{L}{4\pi}\oint d\sigma\, \cos\theta\,\dot{\phi} - \frac{\pi}{2L}\oint d\sigma\big((\theta')^2 + \sin^2\theta(\phi')^2\big).$$

Equations of motion

$$\dot{\phi} = (2\pi/L)^2\big(\cos\theta(\phi')^2 - \csc\theta\,\theta''\big),$$

$$\dot{\theta} = (2\pi/L)^2\big(2\cos\theta\,\theta'\phi' + \sin\theta\,\phi''\big).$$

Momentum P, energy $\tilde{E}$, spin (above vacuum)

$$P = \frac{1}{2}\oint d\sigma(1-\cos\theta)\phi',$$

$$\tilde{E} = \frac{\pi}{2}\oint d\sigma\big((\theta')^2 + \sin^2\theta(\phi')^2\big),$$

$$\alpha = \frac{1}{4\pi}\oint d\sigma(1-\cos\theta).$$

Rational Solution

Vacuum solution: String collapsed at north pole of S^2

$$\theta(\sigma,\tau)=0,\qquad \alpha=\tilde{E}=P=0.$$

Simplest non-trivial solution: Circular string at latitude θ_0 with n windings

$$\theta(\sigma,\tau)=\theta_0,\qquad \phi(\sigma,\tau)=n\sigma+(2\pi/L)^2 n^2\cos\theta_0\,\tau.$$

Momentum & energy as functions of spin α and mode number n

$$\alpha=\tfrac{1}{2}(1-\cos\theta_0),\qquad P=2\pi n\alpha,\qquad \tilde{E}=4\pi^2 n^2\alpha(1-\alpha).$$

Semiclassical quantisation of LL model: towards Heisenberg chain

- Fluctuations: quantum states (HO's) in the vicinity of classical solution.
- Energy shift: quantum correction to classical energy. Sum of HO's.

Fluctuations

Perturb the classical solution with Fourier mode number m

$$\theta(\sigma,\tau) = \theta_0 + \epsilon\theta_+(\tau)e^{ik\sigma} + \epsilon\theta_-(\tau)e^{-ik\sigma} + \epsilon^2\theta_0(\tau),$$
$$\phi(\sigma,\tau) = \phi_0(\sigma,\tau) + \epsilon\phi_+(\tau)e^{ik\sigma} + \epsilon\phi_-(\tau)e^{-ik\sigma} + \epsilon^2\phi_0(\tau).$$

Expand Lagrange function to $\mathcal{O}(\epsilon^2)$:
Two coupled HO's with position $\phi_\pm$ and momentum $\theta_\pm$ variables.

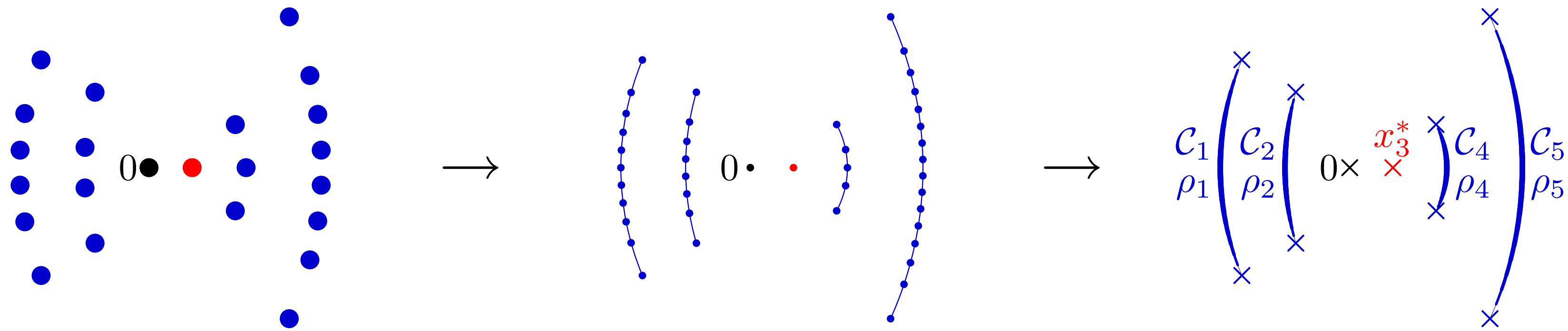
Normalise to one unit in phase space:

$$\delta\alpha = L^{-1},$$
$$\delta P = 2\pi(n+k)L^{-1},$$
$$\delta\tilde{E} = (2\pi)^2 L^{-1}\left(n(n+2k)(1-2\alpha) + k^2\sqrt{1-4n^2\alpha(1-\alpha)/k^2}\right)\ldots.$$

Instability when $2n\sqrt{\alpha(1-\alpha)} > k = 1$ (i.e. for large α and n).

Thermodynamic Limit of Bethe Equations

Roots $u_k = Lx_k$ condense on contours $\mathcal{C}_a$ (or remain single x_a^*): [NB, Minahan Staudacher Zarembo]

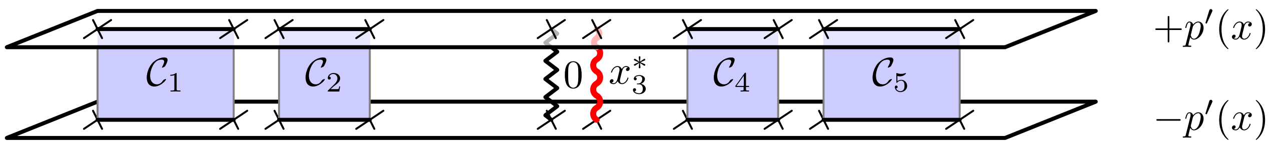


Density of roots $\rho(x)$ specifies discontinuity of quasi-momentum $p(x)$

$$p(x) = \frac{1}{2x} + \sum_a \int_{\mathcal{C}_a} \frac{dy\,\rho_a(y)}{y-x}\,.$$

Derivative $\pm p'(x)$ defines spectral curve. [Kazakov, Marshakov Minahan, Zarembo]

Two Riemann sheets connected by branch cuts $\mathcal{C}_a$.



Same spectral curve from classical LL model. [Kazakov, Marshakov Minahan, Zarembo]

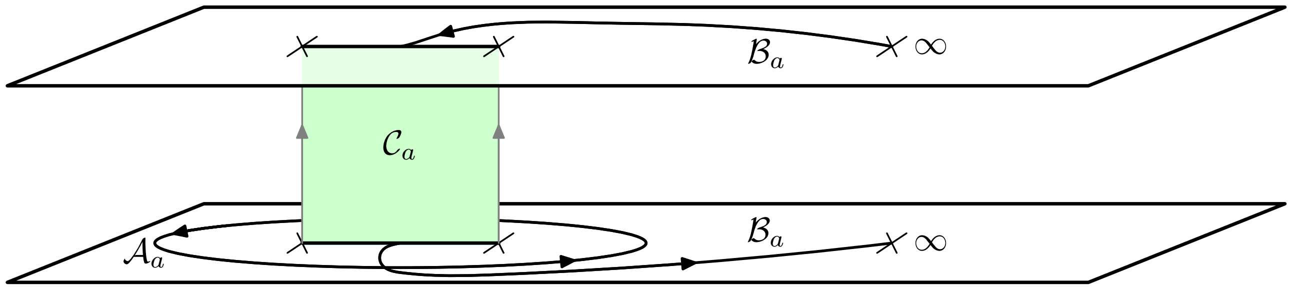
Small Filling

Moduli of a cut: mode number $n_a \in \mathbb{Z}$ and partial filling $\alpha_a \in \mathbb{R}$.

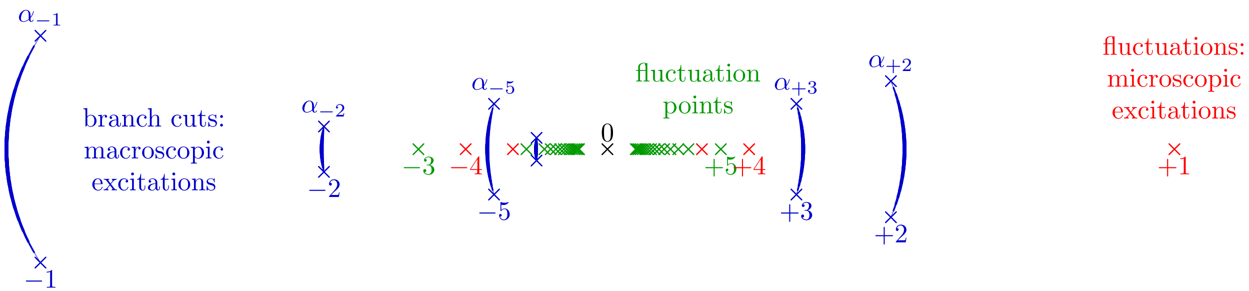
$$0 = \oint_{\mathcal{A}_a} dp,$$

$$n_a = \frac{1}{2\pi}\int_{\mathcal{B}_a} dp,$$

$$\alpha_a = \frac{1}{2\pi i}\oint_{\mathcal{A}_a} x\,dp.$$



General configuration for small enough fillings (stringy modes)



Density of Bethe roots small (no Bethe strings), weakly interacting modes.

From Small to Large Filling

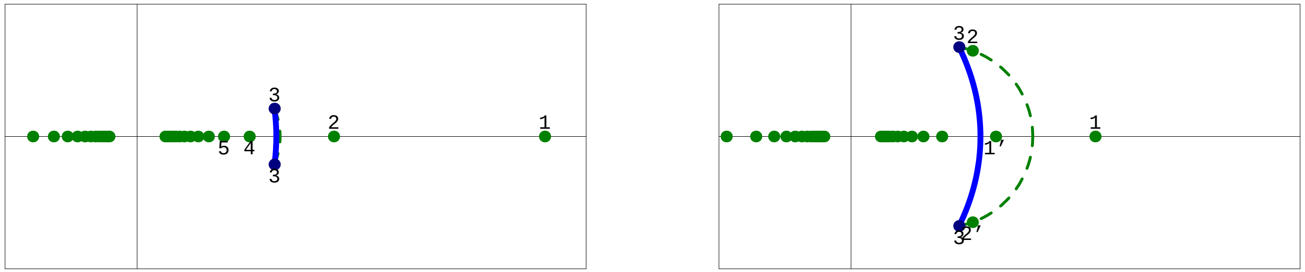
One-Cut Solution

What if the filling is large? What about unstable modes? [Bargheer, NB to appear]

Consider simplest solutions with one branch cut. Quasi-momentum

$$p(x) = \pi n + \frac{1 - 2\pi n x}{2x}\sqrt{1 + \frac{8\pi n \alpha x}{(1 - 2\pi n x)^2}}\,.$$

A macroscopic cut attracts nearby fluctuation points and cuts.



Two interesting situations: [NB, Tseytlin Zarembo] [Hernández, López Periáñez, Sierra] [NB Freyhult]

- Fluctuation point crosses branch cut: Unit density & Bethe equations.
- Fluctuation points meet and drift off into complex plane: Instability.

Two-Cut Solution

Consider small cut instead of fluctuation. Need two-cut solution.

Symmetric two-cut solution known (branch points $\pm a_0, \pm b_0$) [NB, Minahan Staudacher Zarembo]

$$p_0(z) = -\frac{\Delta n\, a_0}{z}\sqrt{\frac{a_0^2(b_0^2-z^2)}{b_0^2(a_0^2-z^2)}}\,\Pi\Big(\frac{qz^2}{z^2-a_0^2}, q\Big)$$

General two-cut solution by Möbius transformation $p_0(\mu(x))$

$$z = \mu(x) = \frac{tx+u}{rx+s}.$$

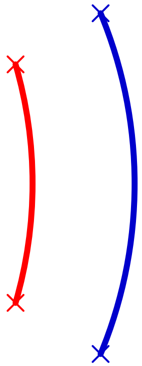
Transformed function $p_0(\mu(x))$ has faulty pole at $x = \mu^{-1}(0) \neq 0$.
Shift pole at $z = 0$ to $x = 0$ (i.e. $z = \mu(0) = u/s$) with $p(x) = p_{u/s}(\mu(x))$

$$p_c(z) = p_0(z) - \frac{\Delta n\,(a_0^2-z^2)\,\mathrm{K}(q)}{a_0 z(z/c-1)}\sqrt{\frac{a_0^2(b_0^2-z^2)}{b_0^2(a_0^2-z^2)}}.$$

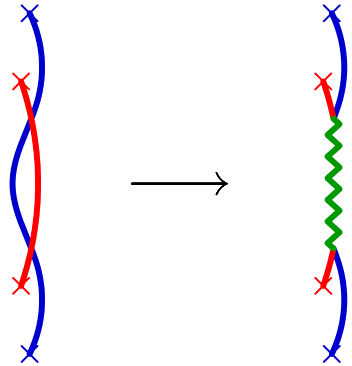
Two Cuts Colliding

Consider two cuts with nearby mode numbers. Cuts attract

- cuts disjoint at small filling

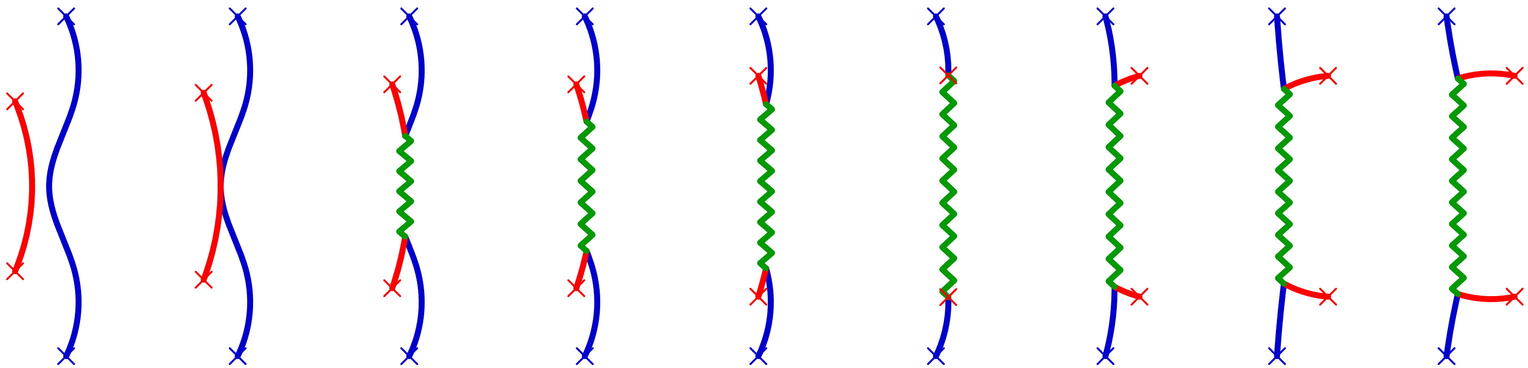


- cuts overlap at larger fillings



condensate with density $\Delta u = i$ forms.

Cuts can pass through other cuts:

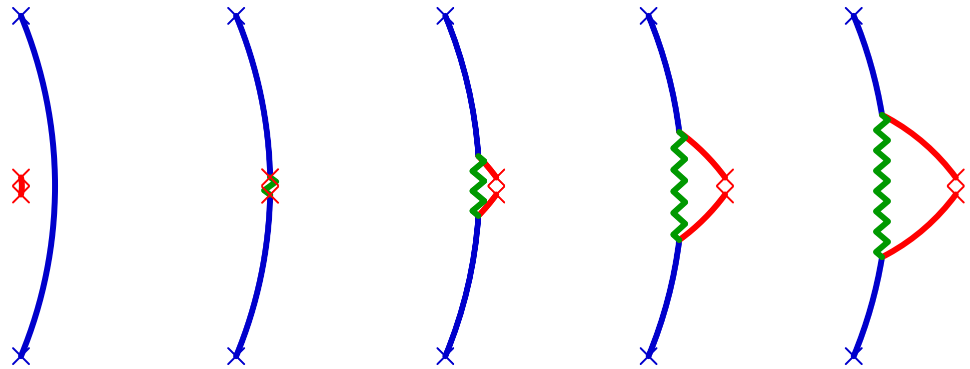


Mode number of passing cut changed: $n_2 \to 2n_1 - n_2$.

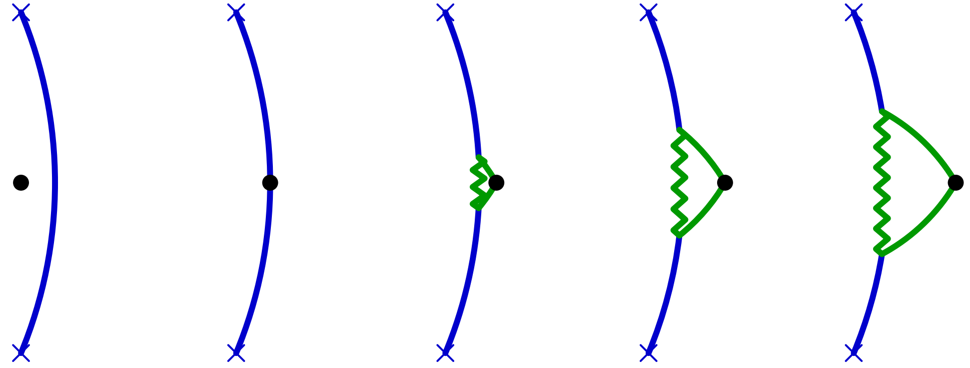
Loop Formation

A fluctuation point should be smoothly connected to a small cut.

Consider a very small cut passing through a macroscopic cut:



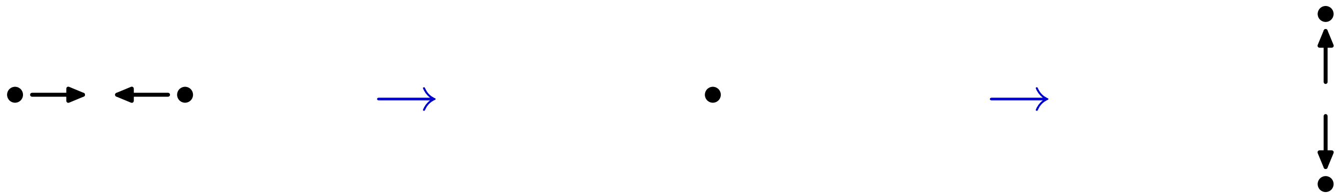
For a fluctuation point the loop closes:



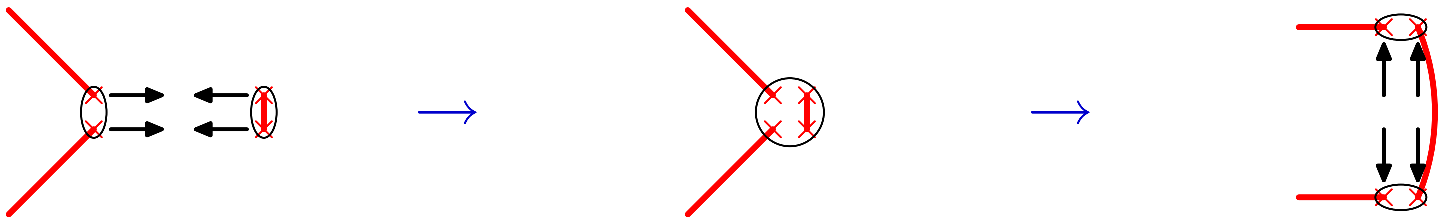
One-loop solution becomes a genus-0 degenerate case of two cuts.

Instability

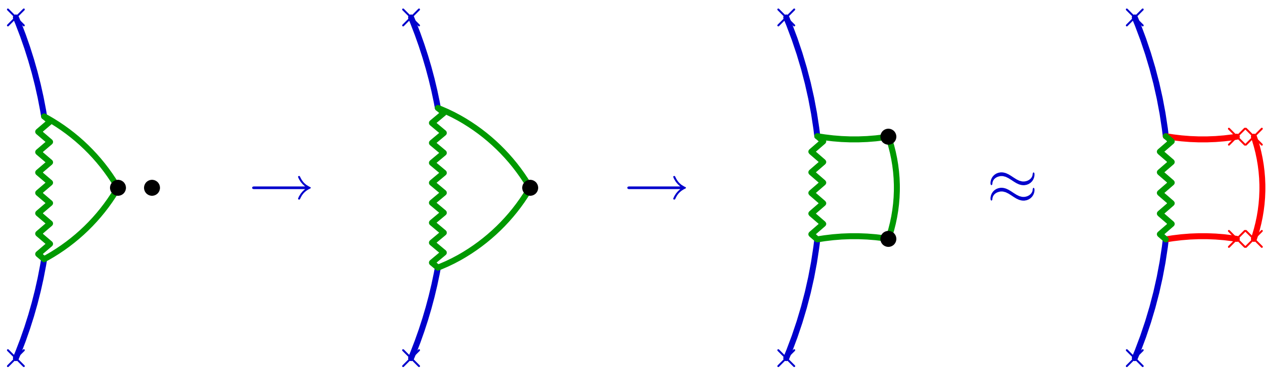
When fluctuation points meet, they drift off into the complex plane:



To understand what happens, consider again small cuts:



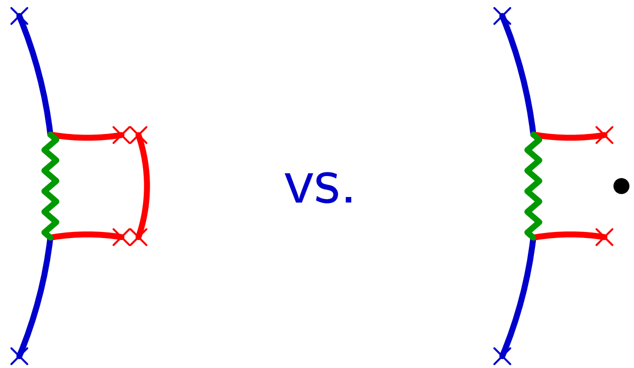
Pairing of branch point reorganises. For fluctuation points the loop closes



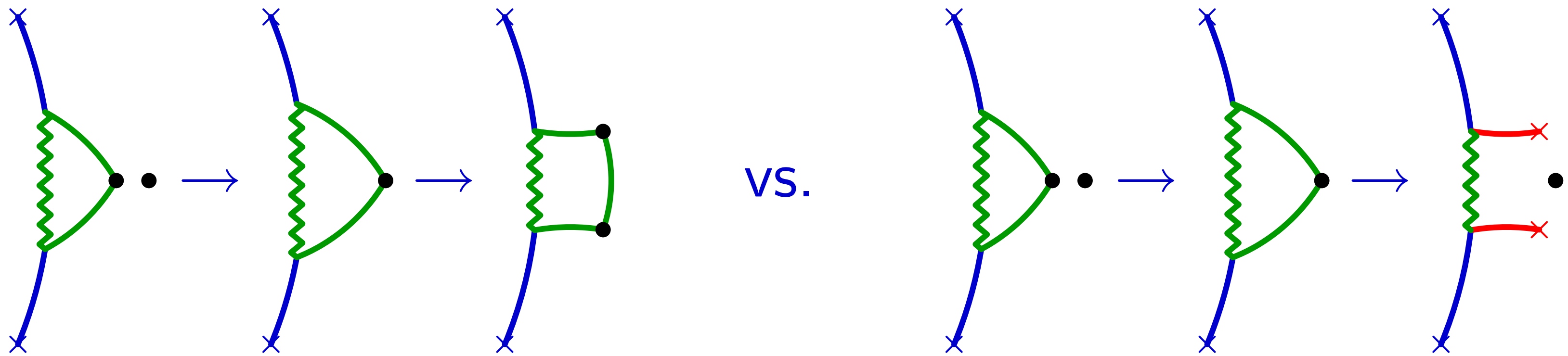
Unstable one-cut solution becomes a degenerate case of three cuts.

Phase Transition

Beyond instability: Solution with lower energy for empty third cut:

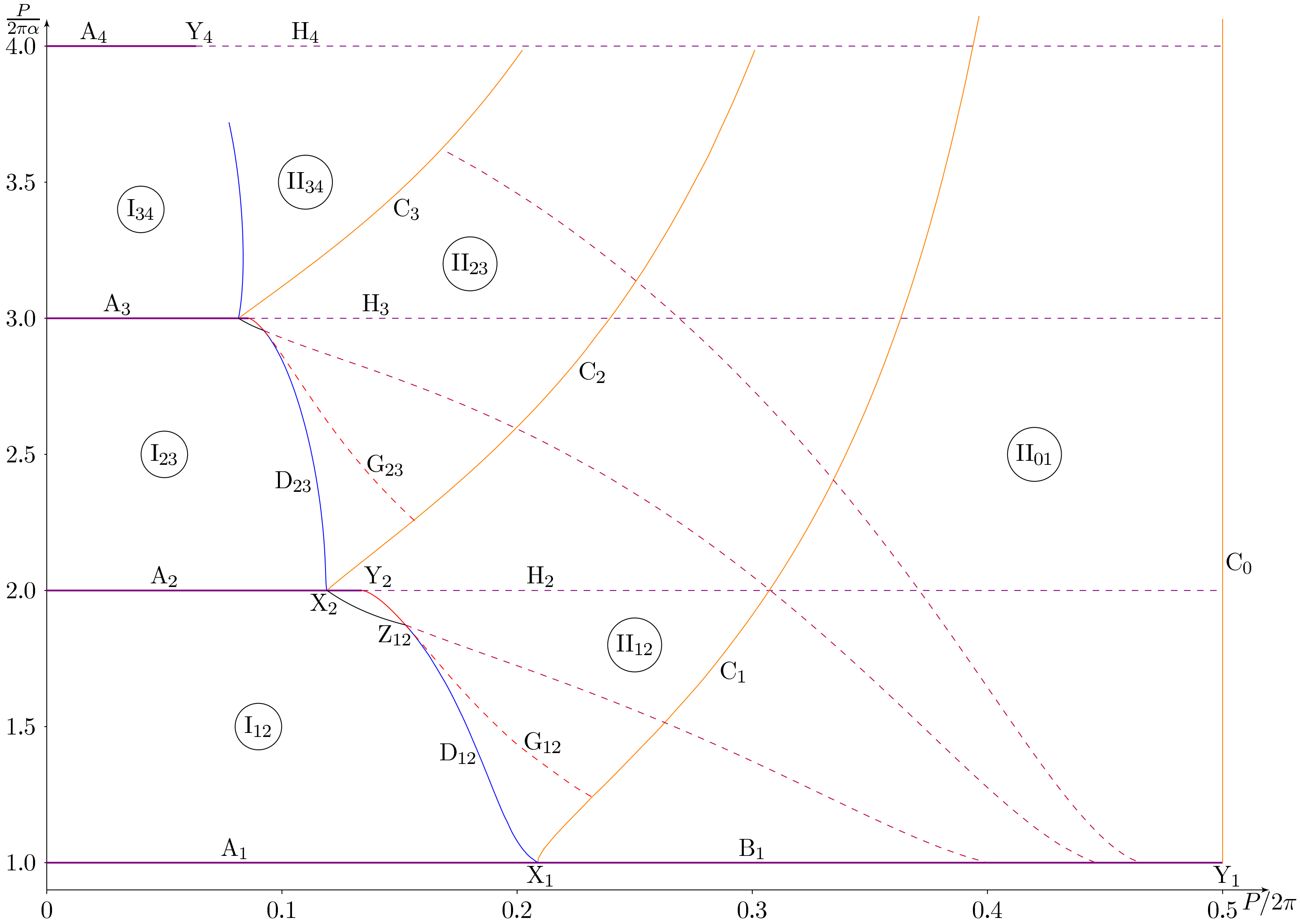


How to continue one-cut solution beyond instability point?



Use "generic" two-cut solution instead of degenerate three-cut solution.
Equivalent to Douglas-Kazakov phase transition for YM on S^2. [Douglas Kazakov]

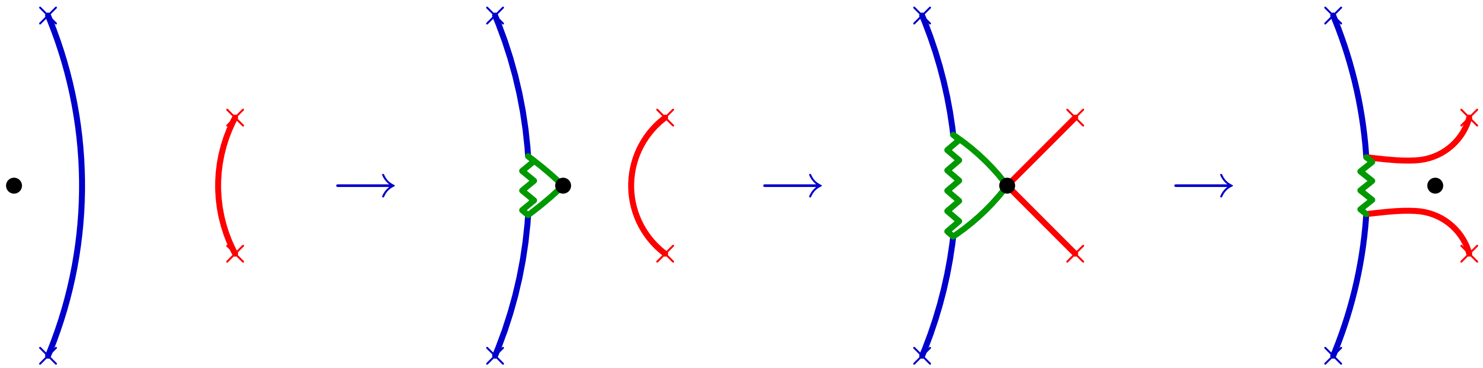
Phase Space for Consecutive Mode Numbers



Everything connected. B/R boundary: $n = 1$ and imaginary axis.

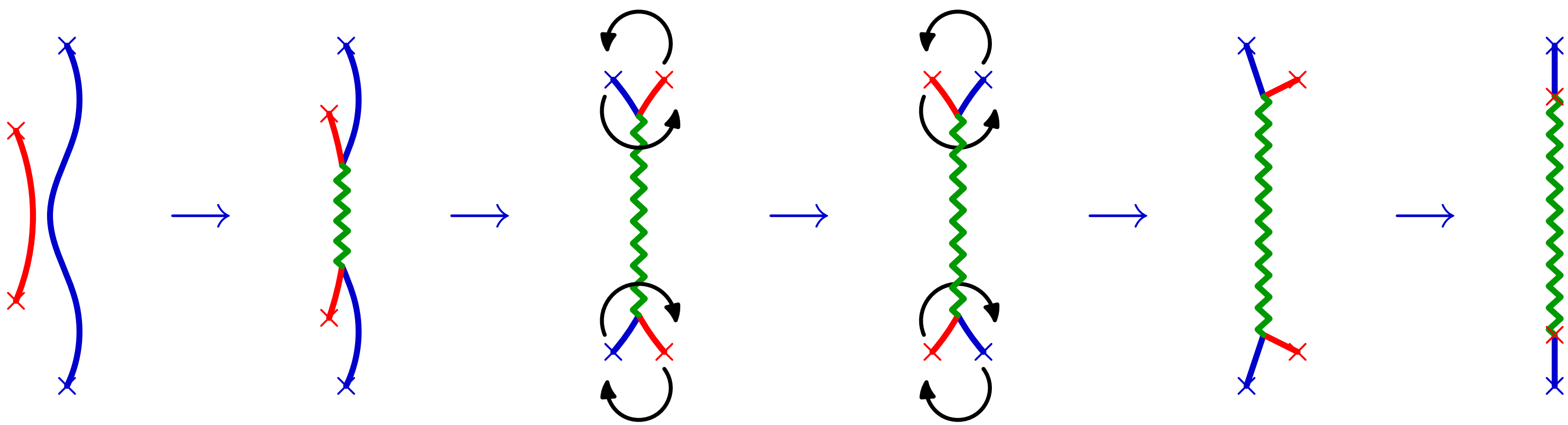
Two Peculiarities

What happens near small corner XYZ of parameter space?



A fluctuation loop joins with a macroscopic cut.

Now increase total filling α while keeping total momentum P fixed.



Cuts join, spin down and approach the imaginary axis at $P = \pi$, $n = 1$.

Conclusions

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⋆ **Spectrum of AdS/CFT**

- Planar asymptotic spectrum described by Bethe equations.
- Asymptotic Bethe equations & strong/weak coupling interpolation.
- Full agreement with AdS/CFT! Several predictions tested (cusp energy).

⋆ **Thermodynamic Limit of the Heisenberg Ferromagnet**

- Equivalent to Landau Lifshitz model.
- Integrability: Spectrum described by spectral curves.
- Macroscopic excitations are cuts. Moduli: mode number and filling.

⋆ **Stability of Spectral Curves**

- Phase space of two-cut solutions investigated.
- Cuts can join and form condensates. Cuts can pass through other cuts.
- Phase transition from one-cut to two-cut solutions at large filling.

- Interactions of more than two cuts?