Geometric Phases in String Theory

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Goals of the Project

- Compute non-Abelian Berry’s Phase in strongly interacting QM systems
  - + to appear...
  - Work done in collaboration with Chris Pedder and David Tong

- Find applications to condensed-matter systems and/or topological quantum computation
Contents of the Talk

- Review of Geometric Phase
- Extension to non-Abelian Geometric Phase
- (2,2): briefly
- (4,4): Yang Monopole and D0-D4 system
- Overview + Outlook on (8,8) model
- Conclusions
Berry Philosophy

- Set system up in a particular energy eigenstate
- Change parameters slowly: Adiabatic theorem means that system clings on to eigenstate

special slide credit: Dave Tong
Review of Berry Phase I

• Canonical Example of Abelian Berry Phase:

• Spin 1/2 in external magnetic field

• Slowly change magnetic field

• Adiabatic Theorem: Cling on to eigenstate

• Quantum Evolution gives law of parallel transport

$$H = \vec{B} \cdot \vec{\sigma}$$
Review of Berry Phase II
(the canonical example)

\[ H_{1/2} = \vec{B} \cdot \sigma \]

- Quantization: \[ H_{1/2}|B_\pm\rangle = \pm B|B_\pm\rangle \]
- Pick ground state and normalize to zero energy
  \[ \mathcal{H} = H_{1/2} - |\vec{B}| \]
- Now ask what happens as \( \vec{B} \) is varied
Review of Berry Phase III
(the general picture)

- Induce gauge connection on $\mathbb{R}^3$

$$A = i\langle B_+(t)|d|B_+(t)\rangle$$

$$|B_+(t)\rangle = \exp \left( -i \int^t E_+(t')dt' \right) \exp \left[ i \int_C \langle B_+|d|B_+\rangle \right] |B_+(0)\rangle$$
Induce gauge connection on $\mathbb{R}^3$

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Dynamical phase
Review of Berry Phase III
(the general picture)

• Induce gauge connection on $\mathbb{R}^3$

\[ A = i\langle B_+(t) | d | B_+(t) \rangle \]

\[ |B_+(t)\rangle = \exp \left( -i \int^t E_+(t') dt' \right) \exp \left[ i \int_C \langle B_+ | d | B_+ \rangle \right] |B_+(0)\rangle \]

Dynamical phase  Geometric phase
Review of Berry Phase IV
(back to our example)

- Follow the ground state
  \[ |\Omega\rangle = \mathcal{N} |B_+\rangle \]

- Berry connection \( A = i \langle \Omega | d |\Omega\rangle \) is the Dirac Monopole

- Berry phase then means
  \[ |\Omega\rangle \rightarrow \exp \left( -i \oint \vec{A} \cdot d\vec{B} \right) |\Omega\rangle \]
Non-Abelian Berry Phase

• If there is degeneracy: Instantaneous basis has natural $U(n)$ action. Pick basis $\{|a\rangle\}_{a=1}^n$

$$A_{ab} = i\langle a(t)|d|b(t)\rangle$$

• Berry Holonomy

$$|a\rangle \rightarrow \mathcal{P} \exp \left( -i \oint (A_\mu)_{ab} \, dX^\mu \right) |b\rangle$$
Rest of the Talk

- Demonstrate that supersymmetry and Berry phases are natural companions
- Two (very) different examples of SUSY systems and non-Abelian Berry phases
- Only have time to talk about (4,4) model in detail
- Two different mechanism to ensure degeneracy over range of parameters
- New interpretation in terms of D-brane precession
(2,2) Geometric Phase

- (2,2) SUSY Quantum Mechanics
- Free chiral multiplet: Dirac Monopole
  - More complicated examples: non-renormalisation
- $\mathbb{CP}^1$ Sigma model with potential
  - Multiple vacua: Witten index
  - Non-Abelian Holonomy interpolates between vacua
  - Receives corrections due to BPS instantons
- Result: non-Abelian Berry connection
(4,4) Geometric Phase

- Turn to higher SUSY system
- Naturally has non-Abelian holonomy: states come in degenerate pairs because of Kramers degeneracy
- Natural interpretation in terms of D0-D4 system
Do-D4 system: overview

• R-Symmetry

\[ Spin(5) \times SU(2)_R \cong Sp(2) \times Sp(1) \]

• Lagrangian

\[ L = L_{\text{vec}} + L_{\text{hyper}} + L_{\text{Yuk}} \]
Do-D$_4$ system: overview

• R-Symmetry

$$Spin(5) \times SU(2)_R \cong Sp(2) \times Sp(1)$$

• Lagrangian

$$L = L_{\text{vec}} + L_{\text{hyper}} + L_{\text{Yuk}}$$

$$\{ A, V \} \quad \{ \Phi, \tilde{\Phi} \} \quad \mathcal{W} = \sqrt{2} \mu \Phi \tilde{\Phi}$$
Do-D4 System: Details

\[ L_{\text{vector}} = \frac{1}{2g^2} (\dot{X}^2 + 2i\bar{\Lambda}\dot{\Lambda}) \]

\[ L_{\text{hyper}} = \sum_{i=1}^{N} |D_t \phi_i|^2 + |D_t \tilde{\phi}_i|^2 + i\bar{\Psi}_i D_t \Psi_i - \tilde{X}^2 (|\phi_i|^2 + |\tilde{\phi}_i|^2) \]

\[ -\frac{g^2}{2} (\sum_i |\phi_i|^2 - |\tilde{\phi}_i|^2)^2 - 2g^2 \sum_i \tilde{\phi}_i \phi_i |^2 \]

\[ L_{\text{Yuk}} = -\bar{\Psi} (\tilde{X} \cdot \tilde{\Gamma}) \Psi + \sqrt{2}\bar{\Psi}_\alpha (\phi \Lambda_\alpha + \tilde{\phi}^\dagger J_\alpha^\beta \Lambda^*_\beta) + \text{h.c.} \]
Born-Oppenheimer

- Take $N = 1$ and $g^2 \to 0$

$$H = |\pi|^2 + |\tilde{\pi}|^2 + X^2 \left( |\phi|^2 + |\tilde{\phi}|^2 \right) + \bar{\Psi} \left( \vec{X} \cdot \Gamma \right) \Psi$$

<table>
<thead>
<tr>
<th>State</th>
<th>Multiplicity</th>
<th>$H_F$ Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>0\rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$\Psi_\alpha</td>
<td>0\rangle$</td>
<td>4</td>
</tr>
<tr>
<td>$\Psi_\alpha \Psi_\beta</td>
<td>0\rangle$</td>
<td>6</td>
</tr>
<tr>
<td>$\Psi_\alpha \Psi_\beta \Psi_\gamma</td>
<td>0\rangle$</td>
<td>4</td>
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<tr>
<td>$\Psi_\alpha \Psi_\beta \Psi_\gamma \Psi_\delta</td>
<td>0\rangle$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: The Fermionic Hilbert Space.
QM Calculation I: Yang

- Ground state undergoes no Berry phase

- 1st excited state $\bar{\Psi}_\alpha|0\rangle$

- pick basis $P_- \lambda_\alpha \bar{\Psi}_\alpha|0\rangle \rightarrow \{|1\rangle, |2\rangle\}$

- Berry connection

$$ (A_\mu)_{ab} = \frac{-X^\nu}{2X(X - X_5)} \eta^{m}_{\mu\nu} \sigma^m_{ab}, \quad A_5 = 0 $$

- Yang Monopole: $S^7 \xrightarrow{S^3} S^4$

- Second Chern Number $C_2 = -1$
QM Calculation II: Yin

- 1st excited state $\star \Psi_{\alpha} |0\rangle$

- pick basis $\lambda_{\alpha} \star \Psi_{\alpha} |0\rangle \rightarrow \{|3\rangle, |4\rangle\}$

- Berry connection

$$\left( \tilde{A}_{\mu} \right)_{ab} = \frac{-X^{\nu}}{2X(X - X_{5})} \bar{\eta}^{m}_{\mu\nu} \sigma^{m}_{ab}, \ A_{5} = 0$$

- Yin Monopole: Second Chern Number

$$C_{2} = 1$$
QM Calculation II: Yin

- 1st excited state: $\star \bar{\Psi}_\alpha |0\rangle$

- pick basis: $\lambda_\alpha \star \bar{\Psi}_\alpha |0\rangle \rightarrow \{|3\rangle, |4\rangle\}$

- Berry connection:

\[
\left(\tilde{A}_\mu\right)_{ab} = \frac{-X^\nu}{2X(X - X_5)}\bar{\eta}^m_{\mu\nu}\sigma^m_{ab}, \ A_5 = 0
\]

- Yin Monopole: Second Chern Number

$$C_2 = 1$$
Yin & Yang

• Putting the two together gives

\[ \Omega_\mu = -\frac{X^\nu}{2X(X - X_5)} \left( \begin{array}{c|c} \eta_{\mu\nu}^m \sigma^m & 0 \\ \hline 0 & \bar{\eta}_{\mu\nu}^m \sigma^m \end{array} \right) \]

• Can undo singular gauge
• Putting the two together gives

\[ \Omega_\mu = -\frac{X^\nu}{2X(X - X_5)} \left( \begin{array}{cc} \eta_{\mu\nu}^m & \sigma^m \\ 0 & \overline{\eta}_{\mu\nu}^m \sigma^m \end{array} \right) \]

• Can undo singular gauge

\[ \Omega_\mu = \frac{X^\nu}{X^2} \Gamma_{\nu\mu} \]
**Effective Action Perspective I**

*(1-loop in d=0+1)*

- Integrate out hypermultiplet

\[ + \quad \cdots \quad \cdots = - \frac{k^2 \delta X(k) \delta X(-k)}{8X^3} \]

- Effective Action

\[ L_{\text{bosonic}} = f(X) \dot{X}^2, \quad f(X) = \frac{1}{2g^2} + \frac{N}{4X^3} \]
**Effective Action Perspective II**

*(strong coupling Berry phase)*

- Complete effective Lagrangian contains

\[ f(X) \left( \dot{X}^2 + \frac{g^2 N}{X^3} \right) \]

- Focus on throat region, where \( g^2 N \gg X^3 \)

- Form of \( L_{\text{eff}} \) protected by non-renormalisation

- Correct description of strong-coupling Higgs-branch dynamics (Berkooz, Verlinde)
Effective Action Perspective III

(strong coupling Berry phase)

- Classical Spin precession = Geometric phase of strongly coupled quantum system

- Read off the spin connection

\[ \Omega_\mu = \frac{3}{2} \frac{X^\nu}{X^2} \Gamma_{\nu\mu} \]

- Suggests new method to compute Berry phase

- Coefficient differs from weak-coupling result: Smooth interpolation or level crossings?
What the Do saw I
(Closed string perspective)

- Treat Do-brane as a probe

\[ ds^2 = \frac{1}{\sqrt{f(R)}} dx_\mu dx^\mu + \sqrt{f(R)} d\vec{R} \cdot d\vec{R} \]

\[ f(R) = 1 + \pi g_s N \frac{(\alpha')^{3/2}}{R^3} \]

- Expand DBI action to quadratic order
What the Do saw II
(Closed string perspective)

• SUSY completion (to this order)

\[ L = f(X) \left( \ddot{X}^2 + i(\bar{\Lambda} \dot{\Lambda} + \dot{\bar{\Lambda}} \Lambda) + \frac{1}{2} \left( f,_{\mu\nu} - \frac{1}{2} f,_{\mu} f,_{\nu} \right) + \frac{1}{2} \left( \bar{\Lambda} \Gamma_{\mu\nu} \Lambda + \bar{\Lambda} \Gamma_{\mu} \Lambda \bar{\Lambda} \Gamma_{\nu} \Lambda \right) \right) \]

• Describes spin precession of a probe particle

• Spin = R-Symmetry representation

D4-brane at weak coupling: \( g^2_{\text{eff}} = e^{2N} \ll 1 \)

DO physics captured as instanton in D4: Higgs branch
QM -> throat of Coulomb branch via Berkooz-Verlinde

D4-brane at strong coupling:
\( 1 \ll g^2_{\text{eff}} \ll N^{4/3} \)

DO in backreacted background of D4s: dynamics same
as above due to non-renormalisation of Diaconescu and Entin
(8,8) Berry Phase

- Consider $U(2)$ D0-brane matrix model
- Separate branes: $U(2) \rightarrow U(1)$
  - break down to Cartan subalgebra
  - excite a stretched string between branes
  - ask what happens as one brane moves around the other
- Berry Phase is Octonionic SO(8) connection
- Last Hopf Map
Division Algebras and SUSY

- Four Hopf Maps, four division algebras, four Berry connections

\[
\begin{align*}
S^1 & \quad & S^3 & \quad & S^7 & \quad & S^{15} \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
S^0 & \cong \mathbb{Z}_2 & S^1 & \quad & S^3 & \quad & S^7 \\
\mathbb{RP}^1 & \quad & \mathbb{CP}^1 & \cong S^2 & \mathbb{HP}^1 & \cong S^4 & \mathbb{OP}^1 & \cong S^8 \\
\uparrow & \quad & \uparrow & \quad & \uparrow & \quad & \uparrow \\
(1, 1) & \quad & (2, 2) & \quad & (4, 4) & \quad & (8, 8) \\
\pm 1 & \quad & U(1) \text{ Dirac} & \quad & SU(2) \text{ Yang} & \quad & SO(8) \text{ Oct}
\end{align*}
\]
Summary

- Berry Phase is natural concept in SUSY systems
  - N=2: sign flip
  - N=4: Dirac Monopole
  - N=8: Yang Monopole
  - N=16: Octonionic Monopole
- N=4: non-Abelian phase from Witten index
- N=8: Berry phase from gravitational precession
  - Relation to d=4+1 dimensional gauge theory
  - Instantons
- Look for Applications (aka: “is it good for anything?”)
thanks for listening