# M-theory on singular $G_2$ manifolds

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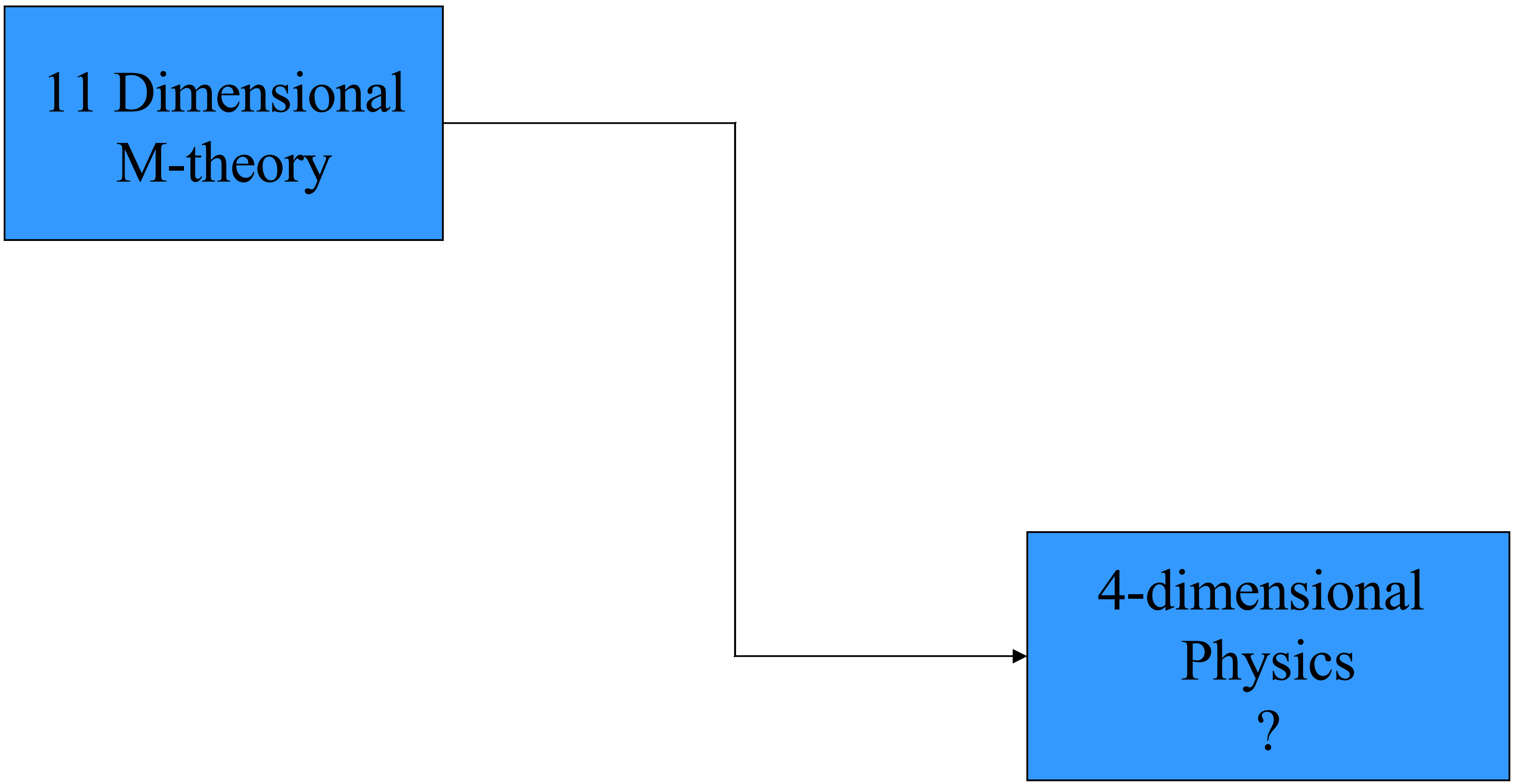
# Outline

- Introduction
  - M-theory phenomenology
  - $G_2$ Spaces: Why we're interested. The problems.
- M-theory on $C^2/Z_N$
  - Motivation: Horava-Witten Theory
  - A similar construction on $C^2/Z_N$
- A $G_2$ Compactification with singularities
  - The theory on a $G_2$ orbifold
  - Wilson Lines and Flux
  - Relationship to N=4 Super Yang-Mills theory
  - Blow-ups and the smooth limit
- Conclusion
  - Results and future directions

# M-theory Compactifications



# M-theory Compactifications

$$M_{11} = M_4 \times X_7$$

11 Dimensional M-theory

4-dimensional Physics ?

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Two approaches:

2. X is a manifold with Boundary and $\partial X$ is Calabi-Yau
2. X is a more general compact 7-dimensional space

4-dimensional Physics ?

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11 Dimensional
M-theory

Two approaches:

2. X is a manifold with Boundary and $\partial X$ is Calabi-Yau

2. X is a more general compact 7-dimensional space

Compact internal spaces

interesting 4-d phenomenology

4-dimensional
Physics
?

# Spinors

- Spinor representations for the 11-d theory decompose as

SO(1,10) - $\underline{32} \longrightarrow \underline{4+8}$

- For N=1 SUSY in 4-d we need 1 covariantly constant spinor on $X_7$

It turns out…

A 7-dimensional $G_2$ space comes with exactly such a structure. Compactification on a $G_2$ space breaks supersymmetry to 1/8 of the original amount (N=1 in 4-d).

( $\underline{8}_{SO(7)} \longrightarrow [\underline{1}+\underline{7}]_{G2}$ )

So, what is a $G_2$ space?…

# Recall, the exceptional Lie-group $G_2$ is…

- 14-real dimensional
- The automorphism group of the octonions
- A $G_2$ space is a 7-dim space with $G_2$ holonomy
- $G_2$ holonomy manifold $\longrightarrow$ Ricci flat

# Properties of $G_2$ spaces

- The space comes equipped with a smooth 3-form, $\varphi_{mnp}$, which is isomorphic to the "flat" 3-form

$$\phi_0 = \left\{ dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \right\}$$

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- This generates an associated metric and a single globally defined spinor

$$g_{mn} = \phi_{mlp} \phi_{nqr} \phi_{stu} \varepsilon^{lpqrstu} \qquad \phi_{mnp} = i \bar{\eta} \gamma_{mnp} \eta$$

- The pair $(\varphi_0, g)$ together form what is called a "$G_2$ structure"
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- The pair($\varphi_0$, g) together form what is called a "$G_2$ structure"
- The subgroup of GL(7,R) preserving $\varphi_0$ is $G_2$
- Several systematic attempts have been made to construct $G_2$ spaces. Examples include the work of Joyce (orbifolded tori) and Hitchin (non-compact spaces).
- It is hard to make $G_2$ Spaces…
    - No $G_2$ version of Yau's theorem
    - Generally a complicated object!

# M-theory compactification on a $G_2$ manifold

- Witten, Papadopoulos and Townsend compactified M-theory on a smooth 7-manifold with $G_2$ holonomy. They found the following field content in 4-d…

- Abelian vector multiplets
  - $b_2(X)$ of them, descending from the 3-form of 11-d SUGRA
- Uncharged chiral multiplets
  - $b_3(X)$ of them, descending from the metric moduli of X and the associated axions.

…This is clearly unphysical!

For reasonable 4-dim physics, we need:

- Non-Abelian gauge fields
- Charged chiral matter

# Singularities

## The problem gets harder…

In the late 90's it was found that while smooth $G_2$-spaces are uninteresting, much better things could occur for singular $G_2$-spaces (Witten, Atiyah, Acharya, etc). In the neighborhoods of these singularities it turns out that…

Co-dim 4 Singularities → Non-Abelian Gauge Fields

Co-dim 7 (Conical) Singularities → <u>Charged</u> Chiral Fermions

So, for realistic physics, we need to construct $G_2$ spaces in which the co-dimension 4 singular locus intersects a conical singularity…this is hard to do!

# The goal…

The current goal of M-theory phenomenology is to produce an effective action for M-theory in the neighborhood of intersecting co-dimension 4 and 7 singularities….

What are the corrections to 11-dimensional supergravity?
What are the properties of the effective 4-dimensional theory?

To answer this question, we turn first to co-dimension four singularities…

# Co-dimension 4 Singularities

We are lead to the idea of singular spaces in M-theory through dualities with heterotic strings.

M-Theory compactified on K3 = Heterotic string theory compactified on a 3-Torus

In particular, we are interested in compactifying 11-d supergravity on a space with orbifold singularities of the form,

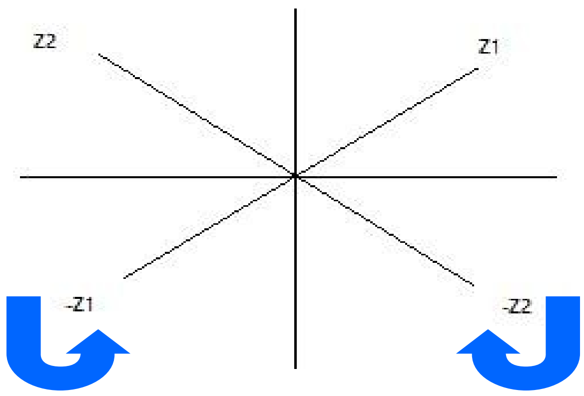
$$C^2/\Gamma_{ADE} \times B_3 \times M_4$$

Where $\Gamma_{ADE}$ is an ADE subgroup of SU(2)

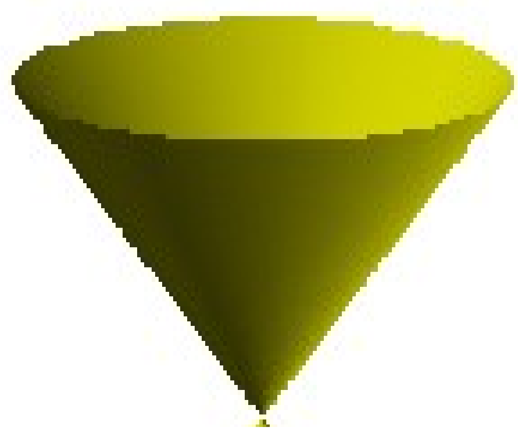
To begin, we'll look at $C^2/Z_N$ type singularities

# $C^2/Z_N$ Orbifolds

$Z_2$ Example:




$Z_2$ symmetry



$Z_N$ symmetry: $(z_1,z_2) \longrightarrow (e^{2\pi i/N} z_1, e\text{-}^{2\pi i/N} z_2)$

Orbifold singularity at $z_1 = z_2 = 0$

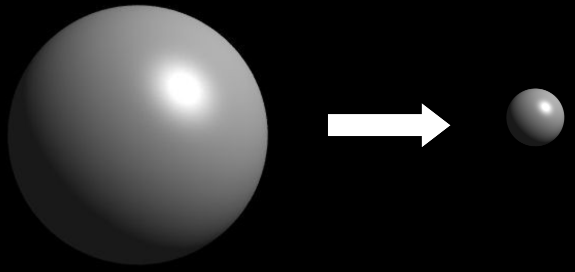
$Z_N$ orbifold $\longrightarrow$ SU(N) gauge fields at $(0,0)\times B_3 \times M_4$

# New states at a singularity

Example: $C^2/Z_2$

We "cut out" the singularity and replace it with an Eguchi-Hanson space, which comes with:

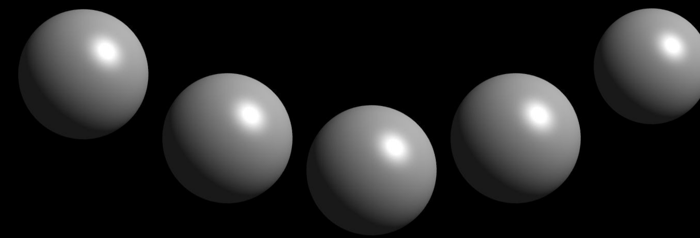
- 2-cycle, *C*, at the origin
- Associated harmonic form $\omega$



Then there is one U(1) vector, A, arising from the 3-form, $C=A_\wedge\omega$

In the limit that *C* shrinks to zero, two additional states arise from a membrane wrapping the 2-cycle.

# New SU(N) fields - $C^2/Z_N$



Blow up the $C^2/Z_N$ singularities with a chain of (N-1) 2-cycles, (with harmonic 2-forms, $\omega_i$ ) at the origin (i.e. a Gibbons-Hawking space).

- $U(1)^{N-1}$ gauge fields, $A^i$ arise from $C=A^i\wedge\omega_i$

- Non-Abelian part of SU(N) arises from membranes wrapping the 2-cycles.

Different origin for Abelian and non-Abelian parts Of SU(N)

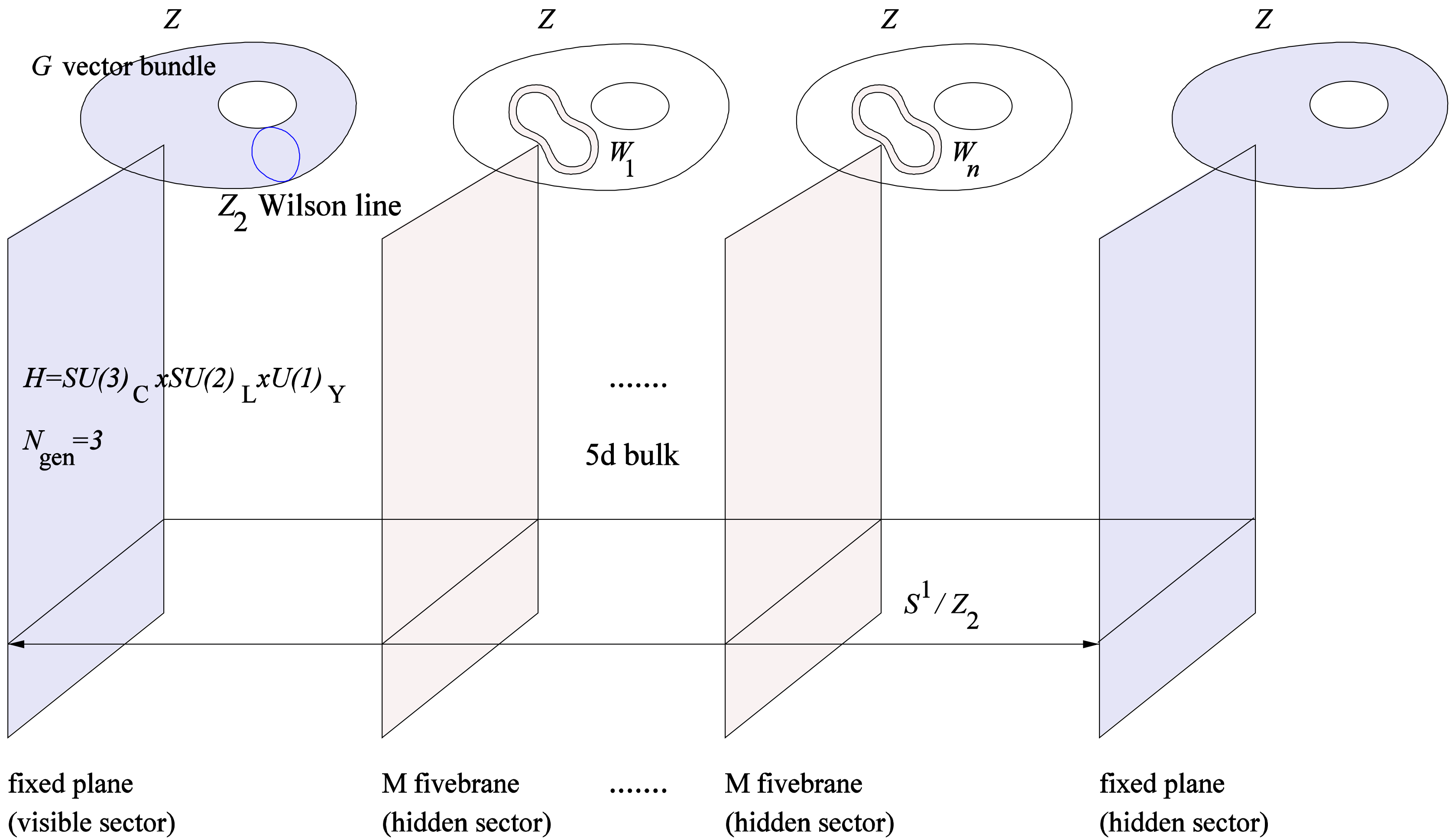
# M-Theory Inspiration: Horava-Witten Theory

Horava and Witten propose that the strong coupling limit of the 10-dim $E_8 x E_8$ heterotic string is 11-dim M-theory compactified on

$$R^{1,9} \times S^1/Z_2$$

With the gauge fields entering via 10-d vector multiplets propagating only on the boundary of spacetime.

The new states in M-theory appeared in the form of 2 $E_8$ Super YM multiplets, located on the two 10-d fixed planes of the orbifold.



Z
Z
Z
Z
G vector bundle
W 1
W n
Z 2 Wilson line
H=SU(3) C xSU(2) L xU(1) Y
N gen =3
.......
5d bulk
S 1 / Z 2
fixed plane
(visible sector)
M fivebrane
(hidden sector)
.......
M fivebrane
(hidden sector)
fixed plane
(hidden sector)

# This implies something interesting…

There must exist a supersymmetric coupling of 10-d vector multiplets on the orbifold fixed plane to the 11-d supergravity multiplet propagating in the bulk!

Horava and Witten explicitly construct such a theory in the following steps…

- Require 11-d SUGRA to be consistent with orbifolding
- Impose conditions from anomaly cancellation
- Add global $E_8$ multiplets to the orbifold fixed planes
- Apply Noether procedure to get a locally supersymmetric theory.

# Brane/Bulk Coupled Theory

$$S_{HW} = \frac{1}{\kappa^2} \int_{M^{11}} dx^{11} \sqrt{-g} \left( R + \ldots \right) + \frac{1}{\lambda^2} \int_{M^{11}} \delta(x^{11}) \left( dx^{11} \sqrt{-g} \left( trF^2 + \ldots \right) \right)$$

Horava-Witten theory proves to be interesting:

Phenomenology, Newton's constant, Gluino condensation as susy breaking, domain walls, etc.

What about the same approach to M-theory on other orbifolds?

# the question:

Can we explicitly write down 11-d SUGRA on the orbifold $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$ coupled to 7-d SU(N) Super-Yang-Mills theory located on the orbifold fixed plane $\mathbb{R}^{1,6} \times \{0\}$?

Further, can such a construction be directly used in a $G_2$ compactification? That is, can we find the structure of low-energy M-theory near a $\mathbb{C}^2/\mathbb{Z}_N$ singularity embedded into a $G_2$ space?

# Overview of the singular M-theory construction:

Constrain 11-d SUGRA to be consistent with orbifolding

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"lift up" to 11-d

11-d SUGRA on $C^2/Z_N$ coupled to 7-d SU(N) SYM

# Review of Einstein Yang-Mills Theory in 7-d (what we're aiming for...)

- Field content: $(g_{\mu\nu}, C_{\mu\nu\rho}, A_{\mu}{}^{i}{}_{j}, \sigma, \psi^i_\mu, \chi^i)$

Gravity multiplet

Gauge multiplet $(A^a_\mu, \varphi^{ai}{}_j, \lambda^{ai})$

- R-symmetry index, i=1,2
- Gauge group G, a=1,…M
- The scalars, $\varphi^{ai}{}_j$ parametrize the coset

  SO(3,M)/SO(3)x SO(M)
- Symplectic Majorana Spinors

# 11-d Supergravity on $C^2/Z_N$

$$\begin{aligned}\mathcal{S}_{11} &= \frac{1}{\kappa^2}\int_{\mathcal{M}_{11}^N} \mathrm{d}^{11}x\sqrt{-g}\Big(\frac{1}{2}R - \frac{1}{2}\bar{\Psi}_M\Gamma^{MNP}\nabla_N\Psi_P - \frac{1}{96}G_{MNPQ}G^{MNPQ} \\ &\qquad - \frac{1}{192}\left(\bar{\Psi}_M\Gamma^{MNPQRS}\Psi_S + 12\bar{\Psi}^N\Gamma^{PQ}\Psi^R\right)G_{NPQR}\Big) \\ &\quad - \frac{1}{12\kappa^2}\int_{\mathcal{M}_{11}^N} C\wedge G\wedge G + \ldots\end{aligned}$$

Has field content: $\left(g_{MN}, \Psi_M, C_{MNP}\right)$

$G = dC$

7+4 Coord split: $x^M = \left(x^\mu, y^A\right) = \left(x^\mu, z^p, \bar{z}^{\bar{p}}\right)$

Spinor decomposition: $\Psi = \psi_i(x,y)\otimes\rho^i + \psi_{\bar{j}}(x,y)\otimes\rho^{\bar{j}}$

# The Orbifold action

$$
\begin{aligned}
e_\mu{}^{\underline{\nu}}(x,Ry) &= e_\mu{}^{\underline{\nu}}(x,y),\\
e_A{}^{\underline{\nu}}(x,Ry) &= (R^{-1})_A{}^B e_B{}^{\underline{\nu}}(x,y),\\
e_\mu{}^{\underline{A}}(x,Ry) &= T^{\underline{A}}{}_{\underline{B}} e_\mu{}^{\underline{B}}(x,y),\\
e_A{}^{\underline{B}}(x,Ry) &= (R^{-1})_A{}^C T^{\underline{B}}{}_{\underline{D}} e_C{}^{\underline{D}}(x,y),\\
C_{\mu\nu\rho}(x,Ry) &= C_{\mu\nu\rho}(x,y),\\
C_{\mu\nu A}(x,Ry) &= (R^{-1})_A{}^B C_{\mu\nu B}(x,y), \quad \text{etc.}\\
\Psi_{\mu i}(x,Ry) &= \Psi_{\mu i}(x,y),\\
\Psi_{\mu\bar{i}}(x,Ry) &= S_{\bar{i}}{}^{\bar{j}}\Psi_{\mu\bar{j}}(x,y),\\
\Psi_{Ai}(x,Ry) &= (R^{-1})_A{}^B \Psi_{Bi}(x,y),\\
\Psi_{A\bar{i}}(x,Ry) &= (R^{-1})_A{}^B S_{\bar{i}}{}^{\bar{j}}\Psi_{B\bar{j}}(x,y).
\end{aligned}
$$

Bi-linear

invariant

not invariant

Where $(R^p{}_q) = e^{2i\pi/N}1_2$ $\quad (R^{\bar{p}}{}_q) = (R^p{}_{\bar{q}}) = 0$ $\quad (T^{\underline{p}}{}_{\underline{q}}) = e^{2i\pi/N}1_2$

$(R^{\bar{p}}{}_{\bar{q}}) = e^{-2i\pi/N}1_2$ $\quad (S_{\underline{i}}{}^{\underline{j}}) = e^{2i\pi/N}1_2$ $\quad (T^{\underline{\bar{p}}}{}_{\underline{\bar{q}}}) = e^{-2i\pi/N}1_2$

# Different 7-d field content for the N=2 and N>2 cases

SO(3,1)/SO(3) coset

U(1) Gauge fields

For N>2 $\quad g_{\mu\nu},\ e_p^{\ \underline{q}},\ C_{\mu\nu\rho},\ C_{\mu p\bar{q}},\ \Psi_{\mu i},\ (\Gamma^p\Psi_p)_i$ and $(\Gamma^{\bar{p}}\Psi_{\bar{p}})_i$

For N=2 $\quad g_{\mu\nu},\ e_A^{\ \underline{B}},\ C_{\mu\nu\rho},\ C_{\mu AB},\ \Psi_{\mu i}$ and $\Psi_{A\bar{i}}$

SO(3,3)/SO(3)xSO(3) coset

$U(1)^3$ Gauge fields

The two cases lead to different structure

# Field Identifications

For $Z_N$

$$
\begin{aligned}
\sigma &= \frac{3}{20}\ln\det g_{AB}, \\
\tilde{g}_{\mu\nu} &= e^{\frac{4}{3}\sigma} g_{\mu\nu}, \\
\psi_{\mu i} &= \Psi_{\mu i} e^{\frac{1}{3}\sigma} - \frac{1}{5}\Upsilon_\mu \left(\Gamma^A \Psi_A\right)_i e^{-\frac{1}{3}\sigma}, \\
\tilde{C}_{\mu\nu\rho} &= C_{\mu\nu\rho}, \\
\chi_i &= \frac{3}{2\sqrt{5}} \left(\Gamma^A \Psi_A\right)_i e^{-\frac{1}{3}\sigma}, \\
F^I_{\mu\nu} &= -\frac{i}{2}\mathrm{tr}\left(\sigma^I G_{\mu\nu}\right), \\
\lambda_i &= \frac{i}{2}\left(\Gamma^p \Psi_p - \Gamma^{\bar{p}} \Psi_{\bar{p}}\right)_i e^{-\frac{1}{3}\sigma}, \\
\ell_I{}^{\underline{J}} &= \frac{1}{2}\mathrm{tr}\left(\bar{\sigma}_I v \sigma^J v^\dagger\right).
\end{aligned}
$$

$$G_{\mu\nu} \equiv (G_{\mu\nu p\bar{q}}),\ v \equiv (e^{5\sigma/6} e^{\bar{p}}{}_{\underline{q}})$$

And for $Z_2$

$$
\begin{aligned}
F^I_{\mu\nu} &= -\frac{1}{4}\mathrm{tr}\left(T^I G_{\mu\nu}\right), \\
\ell_I{}^{\underline{J}} &= \frac{1}{4}\mathrm{tr}\left(\bar{T}_I v T^J v^T\right),
\end{aligned}
$$

# Reviewing the pieces…
# The ‘component’ Lagrangians

- Begin with…

$$L_{11}$$

- Truncate under orbifold action to…

$$L_7^{(n)}$$

- Add SU(N) Yang-Mills

$$L_{SU(N)}$$

- Complete theory is M-theory on $C^2/Z_N$

$$L_{11}+\delta^{(4)}(L_{SU(N)}- L_7^{(n)})$$

# 11-d SUGRA $\longrightarrow$ U(1)$^n$ EYM in 7-d

$$
\begin{aligned}
\mathcal{L}_7^{(n)} = \frac{1}{\kappa_7^2}\sqrt{-\tilde{g}}\Big\{ & \frac{1}{2}R - \frac{1}{2}\bar{\psi}^i_\mu \Upsilon^{\mu\nu\rho}\hat{\mathcal{D}}_\nu \psi_{\rho i} - \frac{1}{4}e^{-2\sigma}\left(\ell_I{}^i{}_j \ell_J{}^j{}_i + \ell_I{}^\alpha \ell_{J\alpha}\right) F^I_{\mu\nu}F^{J\mu\nu} \\
& - \frac{1}{96}e^{4\sigma}\tilde{G}_{\mu\nu\rho\sigma}\tilde{G}^{\mu\nu\rho\sigma} - \frac{1}{2}\bar{\chi}^i\Upsilon^\mu \hat{\mathcal{D}}_\mu \chi_i - \frac{5}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{\sqrt{5}}{2}\left(\bar{\chi}^i\Upsilon^{\mu\nu}\psi_{\mu i} + \bar{\chi}^i\psi^\nu_i\right)\partial_\nu\sigma \\
& - \frac{1}{2}\bar{\lambda}^{\alpha i}\Upsilon^\mu \hat{\mathcal{D}}_\mu \lambda_{\alpha i} - \frac{1}{2}p_{\mu\alpha}{}^i{}_j p^{\mu\alpha j}{}_i - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{\alpha i}\psi^\nu_j\right)p_{\nu\alpha}{}^j{}_i \\
& + e^{2\sigma}\tilde{G}_{\mu\nu\rho\sigma}\Big[\frac{1}{192}\left(12\bar{\psi}^{\mu i}\Upsilon^{\nu\rho}\psi^\sigma_i + \bar{\psi}^i_\lambda \Upsilon^{\lambda\mu\nu\rho\sigma\tau}\psi_{\tau i}\right) + \frac{1}{48\sqrt{5}}\left(4\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi^\sigma_i\right. \\
& \qquad \left. - \bar{\chi}^i\Upsilon^{\mu\nu\rho\sigma\tau}\psi_{\tau i}\right) - \frac{1}{320}\bar{\chi}^i\Upsilon^{\mu\nu\rho\sigma}\chi_i + \frac{1}{192}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu\rho\sigma}\lambda_{\alpha i}\Big] \\
& - ie^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\Big[\frac{1}{4\sqrt{2}}\left(\bar{\psi}^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar{\psi}^{\mu i}\psi^\nu_j\right) + \frac{1}{2\sqrt{10}}\left(\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar{\chi}^i\Upsilon^\mu\psi^\nu_j\right) \\
& \qquad + \frac{3}{20\sqrt{2}}\bar{\chi}^i\Upsilon^{\mu\nu}\chi_j - \frac{1}{4\sqrt{2}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j}\Big] \\
& + e^{-\sigma}F^I_{\mu\nu}\ell_{I\alpha}\Big[\frac{1}{4}\left(2\bar{\lambda}^{\alpha i}\Upsilon^\mu\psi^\nu_i - \bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt{5}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\chi_i\Big] \\
& - \frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}C_{\mu\nu\rho}F^{\tilde{I}}_{\sigma\kappa}F_{\tilde{I}\lambda\tau}\Big\} \, .
\end{aligned}
$$

To the bulk 11-d theory we now want to add SU(N) multiplets on the orbifold fixed plane. So the coset structure of the scalar fields of the 7-d Einstein Yang-Mills theory will become

$SO(3, n+N^2-1)/SO(3) \times SO(n+ N^2 -1)$

Where n=1,3.

Note that the gravity and SU(N) scalars have now become entangled.

# The Action Schematically

$$S_{11-7} = \int_{\mathcal{M}_{11}^N} \mathrm{d}^{11}x \left[\mathcal{L}_{11} + \delta^{(4)}(y^A)\mathcal{L}_{\text{brane}}\right],$$

$$\mathcal{L}_{\text{brane}} = \mathcal{L}_{\text{SU(N)}} - \mathcal{L}_7^{(n)}.$$

## Susy transformations

$$\begin{aligned} \delta_{11} &= {\delta_{11}}^{11} + \kappa^{8/9}\delta^{(4)}(y^A){\delta_{11}}^{\text{brane}} \\ \delta_7 &= {\delta_7}^{\text{SU(N)}}, \end{aligned}$$

Where $\quad {\delta_{11}}^{\text{brane}} = {\delta_{11}}^{\text{SU(N)}} - {\delta_{11}}^{11}$

$\delta_{11}$ - acts on bulk fields

$\delta_7$ - acts on brane fields

${\delta_{11}}^{11}$ - 11-d susy transformations

$\delta_7$ - 7-d susy transformations

# Expansion in coupling constants:

Because the coset structure of the 7-d theory entangles the gravity and scalar fields in the coset,

SO(3, n+N$^2$-1)/SO(3)xSO(n+ N$^2$ -1),

in order to quantitatively understand the corrections, we perform an expansion in the coupling constants:

$$\mathcal{L}_{SU(N)} = \kappa_7^{-2} \left( \mathcal{L}_{(0)} + h^2 \mathcal{L}_{(2)} + h^4 \mathcal{L}_{(4)} + \ldots \right)$$

Where

$$\kappa_7 = \kappa^{5/9} \qquad h = \kappa_7 / g_{\rm YM}$$

and

$$\delta_{11}{}^{SU(N)} = \delta_{11}^{(0)} + h^2 \delta_{11}^{(2)} + h^4 \delta_{11}^{(4)} + \ldots .$$

# We expand to order h²

The coset representative then takes the form

$$L = \begin{pmatrix} \ell + \frac{1}{2}h^2\ell\Phi^T\Phi & m & h\ell\Phi^T \\ h\Phi & 0 & \mathbf{1}_{N^2-1} + \frac{1}{2}h^2\Phi\Phi^T \end{pmatrix}$$

Unlike Horava-Witten, the 7-d theory does not fix the value of $g_{YM}$. However from comparing with IIA D6 branes, (Friedmann and Witten) one finds,

$$g^2_{\mathrm{YM}} = (4\pi)^{4/3}\kappa^{2/3}$$

- The coupled theory is supersymmetric to order $h^2$
- At order $h^4$ we encounter $\delta(0)$ singularity (also present in Horava-Witten theory).

$$
\begin{aligned}
\mathcal{L}_{\mathrm{brane}} = \frac{1}{g_{\mathrm{YM}}^2}\sqrt{-\tilde{g}}\Big\{ & -\frac{1}{4}e^{-2\sigma}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{1}{2}\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j\hat{\mathcal{D}}^\mu\phi^{aj}{}_i - \frac{1}{2}\bar{\lambda}^{ai}\Upsilon^\mu\hat{\mathcal{D}}_\mu\lambda_{ai} - e^{-2\sigma}\ell_I{}^i{}_j\phi_a{}^j{}_iF^I_{\mu\nu}F^{a\mu\nu} \\
& -\frac{1}{2}e^{-2\sigma}\ell_I{}^i{}_j\phi^{aj}{}_i\ell_J{}^k{}_l\phi_a{}^l{}_kF^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}p_{\mu\alpha}{}^i{}_j\phi_a{}^j{}_ip^{\mu\alpha k}{}_l\phi^{al}{}_k \\
& +\frac{1}{4}\phi_a{}^i{}_k\hat{\mathcal{D}}_\mu\phi^{ak}{}_j\left(\bar{\psi}^j_\nu\Upsilon^{\nu\mu\rho}\psi_{\rho i} + \bar{\chi}^j\Upsilon^\mu\chi_i + \bar{\lambda}^{\alpha j}\Upsilon^\mu\lambda_{\alpha i}\right) \\
& -\frac{1}{2\sqrt{2}}\left(\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{\alpha i}\psi^\nu_j\right)\phi_a{}^j{}_i\phi^{ak}{}_lp_{\nu\alpha}{}^l{}_k - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{ai}\psi^\nu_j\right)\hat{\mathcal{D}}_\nu\phi_a{}^j{}_i \\
& +\frac{1}{192}e^{2\sigma}\tilde{G}_{\mu\nu\rho\sigma}\bar{\lambda}^{ai}\Upsilon^{\mu\nu\rho\sigma}\lambda_{ai} + \frac{i}{4\sqrt{2}}e^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\lambda_{aj} \\
& -\frac{i}{2}e^{-\sigma}\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i + 2F^a_{\mu\nu}\phi_a{}^j{}_i\right)\Big[\frac{1}{4\sqrt{2}}\left(\bar{\psi}^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar{\psi}^{\mu i}\psi^\nu_j\right) \\
& \qquad +\frac{3}{20\sqrt{2}}\bar{\chi}^i\Upsilon^{\mu\nu}\chi_j - \frac{1}{4\sqrt{2}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j} + \frac{1}{2\sqrt{10}}\left(\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar{\chi}^i\Upsilon^\mu\psi^\nu_j\right)\Big] \\
& +e^{-\sigma}F_{a\mu\nu}\left[\frac{1}{4}\left(2\bar{\lambda}^{ai}\Upsilon^\mu\psi^\nu_i - \bar{\lambda}^{ai}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt{5}}\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\chi_i\right] \\
& +\frac{1}{4}e^{2\sigma}f_{bc}{}^af_{dea}\phi^{bi}{}_k\phi^{ck}{}_j\phi^{dj}{}_l\phi^{el}{}_i - \frac{1}{2}e^\sigma f_{abc}\phi^{bi}{}_k\phi^{ck}{}_j\left(\bar{\psi}^j_\mu\Upsilon^\mu\lambda^a_i + \frac{2}{\sqrt{5}}\bar{\chi}^j\lambda_i{}^a\right) \\
& -\frac{i}{\sqrt{2}}e^\sigma f_{ab}{}^c\phi_c{}^i{}_j\bar{\lambda}^{aj}\lambda^b_i + \frac{i}{60\sqrt{2}}e^\sigma f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j\Big(5\bar{\psi}^i_\mu\Upsilon^{\mu\nu}\psi_{\nu i} + 2\sqrt{5}\bar{\psi}^i_\mu\Upsilon^\mu\chi_i \\
& \qquad\qquad +3\bar{\chi}^i\chi_i - 5\bar{\lambda}^{\alpha i}\lambda_{\alpha i}\Big) - \frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}\tilde{C}_{\mu\nu\rho}F^a_{\sigma\kappa}F_{a\lambda\tau}\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\rm brane} = \frac{1}{g^2_{\rm YM}}\sqrt{-\tilde g}\Big\{ & -\frac14 e^{-2\sigma} F^a_{\mu\nu}F_a^{\mu\nu} - \frac12 \hat{\mathcal D}_\mu \phi_a{}^i{}_j \hat{\mathcal D}^\mu \phi^{aj}{}_i - \frac12 \bar\lambda^{ai}\Upsilon^\mu \hat{\mathcal D}_\mu \lambda_{ai} - e^{-2\sigma}\ell_I{}^i{}_j \phi_a{}^j{}_i F^I_{\mu\nu}F^{a\mu\nu} \\
& -\frac12 e^{-2\sigma}\ell_I{}^i{}_j\phi^{aj}{}_i \ell_J{}^k{}_l \phi_a{}^l{}_k F^I_{\mu\nu}F^{J\mu\nu} - \frac12 p_{\mu\alpha}{}^i{}_j \phi_a{}^j{}_i p^{\mu\alpha k}{}_l \phi^{al}{}_k \\
& +\frac14 \phi_a{}^i{}_k \hat{\mathcal D}_\mu \phi^{ak}{}_j \left(\bar\psi^j_\nu \Upsilon^{\nu\mu\rho}\psi_{\rho i} + \bar\chi^j\Upsilon^\mu\chi_i + \bar\lambda^{\alpha j}\Upsilon^\mu\lambda_{\alpha i}\right) \\
& -\frac{1}{2\sqrt2}\left(\bar\lambda^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar\lambda^{\alpha i}\psi^\nu_j\right)\phi_a{}^j{}_i \phi^{ak}{}_l p_{\nu\alpha}{}^l{}_k - \frac{1}{\sqrt2}\left(\bar\lambda^{ai}\Upsilon^{\mu\nu}\psi_{\mu j}+\bar\lambda^{ai}\psi^\nu_j\right)\hat{\mathcal D}_\nu\phi_a{}^j{}_i \\
& +\frac{1}{192}e^{2\sigma}\tilde G_{\mu\nu\rho\sigma}\bar\lambda^{ai}\Upsilon^{\mu\nu\rho\sigma}\lambda_{ai} + \frac{i}{4\sqrt2}e^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\bar\lambda^{ai}\Upsilon^{\mu\nu}\lambda_{aj} \\
& -\frac{i}{2}e^{-\sigma}\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i + 2F^a_{\mu\nu}\phi_a{}^j{}_i\right)\Big[\frac{1}{4\sqrt2}\left(\bar\psi^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar\psi^{\mu i}\psi^\nu_j\right) \\
& \qquad +\frac{3}{20\sqrt2}\bar\chi^i\Upsilon^{\mu\nu}\chi_j - \frac{1}{4\sqrt2}\bar\lambda^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j} + \frac{1}{2\sqrt{10}}\left(\bar\chi^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar\chi^i\Upsilon^\mu\psi^\nu_j\right)\Big] \\
& +e^{-\sigma}F_{a\mu\nu}\Big[\frac14\left(2\bar\lambda^{ai}\Upsilon^\mu\psi^\nu_i - \bar\lambda^{ai}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt5}\bar\lambda^{ai}\Upsilon^{\mu\nu}\chi_i\Big] \\
& +\frac14 e^{2\sigma} f_{bc}{}^a f_{dea}\phi^{bi}{}_k\phi^{ck}{}_j\phi^{dj}{}_l\phi^{el}{}_i - \frac12 e^\sigma f_{abc}\phi^{bi}{}_k\phi^{ck}{}_j\left(\bar\psi^j_\mu\Upsilon^\mu\lambda^a_i + \frac{2}{\sqrt5}\bar\chi^j\lambda_i{}^a\right) \\
& -\frac{i}{\sqrt2}e^\sigma f_{ab}{}^c\phi_c{}^i{}_j\bar\lambda^{aj}\lambda^b_i + \frac{i}{60\sqrt2}e^\sigma f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j\Big(5\bar\psi^i_\mu\Upsilon^{\mu\nu}\psi_{\nu i} + 2\sqrt5\bar\psi^i_\mu\Upsilon^\mu\chi_i \\
& \qquad +3\bar\chi^i\chi_i - 5\bar\lambda^{\alpha i}\lambda_{\alpha i}\Big) - \frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}\tilde C_{\mu\nu\rho}F^a_{\sigma\kappa}F_{a\lambda\tau}\Big\}.
\end{aligned}
$$

# Supersymmetry corrections

$$\begin{aligned}
\delta^{\text{brane}}\psi_{\mu i} &= \frac{\kappa_7^2}{g_{\text{YM}}^2}\left\{\frac{1}{2}\left(\phi_{ak}{}^{j}\hat{\mathcal{D}}_\mu\phi^a{}_i{}^k - \phi^a{}_i{}^k\hat{\mathcal{D}}_\mu\phi_{ak}{}^j\right)\varepsilon_j - \frac{i}{15\sqrt{2}}\Upsilon_\mu\varepsilon_i f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j e^\sigma \right.\\
&\left. \qquad + \frac{i}{10\sqrt{2}}\left(\Upsilon_\mu{}^{\nu\rho} - 8\delta_\mu^\nu\Upsilon^\rho\right)\varepsilon_j\left(F^I_{\nu\rho}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i + 2F^a_{\nu\rho}\phi_a{}^j{}_i\right)e^{-\sigma}\right\},\\
\delta^{\text{brane}}\chi_i &= \frac{\kappa_7^2}{g_{\text{YM}}^2}\left\{-\frac{i}{2\sqrt{10}}\Upsilon^{\mu\nu}\varepsilon_j\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i + 2F^a_{\mu\nu}\phi_a{}^j{}_i\right)e^{-\sigma}\right.\\
&\left. \qquad + \frac{i}{3\sqrt{10}}\varepsilon_i f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j e^\sigma\right\},\\
\ell_I{}^i{}_j\delta^{\text{brane}}A^I_\mu &= \frac{\kappa_7^2}{g_{\text{YM}}^2}\left\{\left(\frac{i}{\sqrt{2}}\bar{\psi}^k_\mu\varepsilon_l - \frac{i}{\sqrt{10}}\bar{\chi}^k\Upsilon_\mu\varepsilon_l\right)\phi^{al}{}_k\phi_a{}^i{}_j e^\sigma - \bar{\varepsilon}^k\Upsilon_\mu\lambda^a_k\phi_a{}^i{}_j e^\sigma\right\},
\end{aligned}$$

$$\begin{aligned}
\ell_I{}^\alpha\delta^{\text{brane}}A^I_\mu &= 0,\\
\delta^{\text{brane}}\ell_I{}^i{}_j &= \frac{\kappa_7^2}{g_{\text{YM}}^2}\left\{\frac{i}{\sqrt{2}}\left[\bar{\varepsilon}^k\lambda_{al}\phi^{al}{}_k\phi_a{}^i{}_j\ell_I{}^\alpha + \bar{\varepsilon}^l\lambda_{ak}\phi^{ai}{}_j\ell_I{}^k{}_l - \left(\bar{\varepsilon}^i\lambda_{aj} - \frac{1}{2}\delta^i_j\bar{\varepsilon}^m\lambda_{am}\right)\phi^{al}{}_k\ell_I{}^k{}_l\right]\right\}\\
\delta^{\text{brane}}\ell_I{}^\alpha &= \frac{\kappa_7^2}{g_{\text{YM}}^2}\left\{-\frac{i}{\sqrt{2}}\bar{\varepsilon}^i\lambda^\alpha_j\phi^{aj}{}_i\phi_a{}^l{}_k\ell_I{}^k{}_l\right\},\\
\delta^{\text{brane}}\lambda^\alpha_i &= \frac{\kappa_7^2}{g_{\text{YM}}^2}\left\{\frac{i}{\sqrt{2}}\Upsilon^\mu\varepsilon_j\phi_{ai}{}^j p_\mu{}^{\alpha k}{}_l\phi^{al}{}_k\right\},
\end{aligned}$$

# The Brane Bosonic Theory

$$\begin{aligned}\mathcal{S}_{7,\mathrm{bos}} &= \frac{1}{g_{\mathrm{YM}}^2}\int_{y=0} \mathrm{d}^7x\sqrt{-g}\Big(-\frac{1}{4}H_{ab}F^a_{\mu\nu}F^{b\mu\nu} - \frac{1}{2}H_{aI}F^a_{\mu\nu}F^{I\mu\nu} - \frac{1}{4}(\delta H)_{IJ}F^I_{\mu\nu}F^{J\mu\nu} \\ &\qquad -\frac{1}{2}e^{\tau}\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j\hat{\mathcal{D}}^\mu\phi^{aj}{}_i - \frac{1}{2}(\delta K)^{\alpha j}{}_i{}^{\beta l}{}_k\, p_{\mu\alpha}{}^i{}_j p^\mu{}_\beta{}^k{}_l + \frac{1}{4}D^{ai}{}_j D_a{}^j{}_i\Big) \\ &\quad -\frac{1}{4g_{\mathrm{YM}}^2}\int_{y=0} C\wedge F^a\wedge F_a, \qquad (5\end{aligned}$$

Gauge-Kinetic Function (SU(N))

Gauge-Kinetic Function for gravi-photons

Contributes to D-term potential

$$\begin{aligned} H_{ab} &= \delta_{ab}, \\ H_{aI} &= 2\ell_I{}^i{}_j\phi_a{}^j{}_i, \\ (\delta H)_{IJ} &= 2\ell_I{}^i{}_j\phi_a{}^j{}_i\ell_J{}^k{}_l\phi_a{}^l{}_k, \\ (\delta K)^{\alpha j}{}_i{}^{\beta l}{}_k &= e^{\tau}\delta^{\alpha\beta}\phi_a{}^j{}_i\phi^{al}{}_k, \\ D^{ai}{}_j &= e^{\tau}f^a{}_{bc}\phi^{bi}{}_k\phi^{ck}{}_j. \end{aligned}$$

Where we recall, the bulk fields are coupled in this action through

$$F^I_{\mu\nu} = -\frac{i}{2}\mathrm{tr}\,(\sigma^I G_{\mu\nu}) \qquad \ell_I{}^i{}_j = \frac{1}{\sqrt{2}}\ell_I{}^u(\sigma_u)^i{}_j \qquad D^a = \frac{1}{2}e^{\tau}f^a{}_{bc}[\phi^b,\phi^c]$$

$$\tau = \frac{1}{2}\ln\det g_{AB} \qquad p_{\mu\alpha}{}^i{}_j = \ell^I{}_\alpha\partial_\mu\ell_I{}^i{}_j, \qquad V = \frac{1}{4g_{\mathrm{YM}}^2}\mathrm{tr}\,(D^aD_a)$$

# $G_2$ Compactification

After constructing the 7-dimensional theory, we are now ready to embed our singular neighborhood into a $G_2$ space…

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- We will utilize $G_2$ orbifolds, $T^7/\Gamma$, constructed by dividing a 7-torus, $T^7$, by a discrete symmetry, $\Gamma$, such that the resulting singularities are of co-dimension 4 and A-type. We choose particular symmetry groups such that the singular loci will always be 3-tori, $T^3$.

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- In the neighborhood of a singularity, the $G_2$ space looks like

  $C^2/Z_N \times T^3$.
- While the full 4-dimensional theory will be $\mathcal{N}=1$ supersymmetric, the gauge sub-sectors associated to each singularity have enhanced $\mathcal{N}=4$ supersymmetry.

To begin the compactification, we write the 11-dimensional metric as

$$\mathrm{d}s^2 = \left(\prod_{A=1}^{7} R^A\right)^{-1} g_{\mu\nu} dx^\mu dx^\nu + \sum_{A=1}^{7} \left(R^A \mathrm{d}x^A\right)^2$$

where the $R^A$ are the seven radii of the $T^7$

There exists a $G_2$ structure, a harmonic 3 form associated to the metric above

$$\begin{aligned}\varphi \;&=\; R^1R^2R^3\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + R^1R^4R^5\mathrm{d}x^1\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5 - R^1R^6R^7\mathrm{d}x^1\wedge\mathrm{d}x^6\wedge\mathrm{d}x^7\\ &+R^2R^4R^6\mathrm{d}x^2\wedge\mathrm{d}x^4\wedge\mathrm{d}x^6 + R^2R^5R^7\mathrm{d}x^2\wedge\mathrm{d}x^5\wedge\mathrm{d}x^7 + R^3R^4R^7\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\mathrm{d}x^7\\ &-R^3R^5R^6\mathrm{d}x^3\wedge\mathrm{d}x^5\wedge\mathrm{d}x^6.\end{aligned}$$

where the some of the $R^A$ are related by orbifolding). From this we defin he metric moduli

$$\begin{aligned}&a^0 = R^1R^2R^3, \quad a^1 = R^1R^4R^5, \quad a^2 = R^1R^6R^7, \quad a^3 = R^2R^4R^6,\\ &a^4 = R^2R^5R^7, \quad a^5 = R^3R^4R^7, \quad a^6 = R^3R^5R^6.\end{aligned}$$

Similarly, the 3-form of 11-dim SUGRA can be expanded as

$$\begin{aligned}C \;&=\; \nu^0\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + \nu^1\mathrm{d}x^1\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5 - \nu^2\mathrm{d}x^1\wedge\mathrm{d}x^6\wedge\mathrm{d}x^7 + \nu^3\mathrm{d}x^2\wedge\mathrm{d}x^4\wedge\mathrm{d}x^6\\ &+\nu^4\mathrm{d}x^2\wedge\mathrm{d}x^5\wedge\mathrm{d}x^7 + \nu^5\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\mathrm{d}x^7 - \nu^6\mathrm{d}x^3\wedge\mathrm{d}x^5\wedge\mathrm{d}x^6.\end{aligned}$$

# Field Content from the singularity

In addition to the field content from 11-dim SUGRA, we also have contributions from the 7-dim Einstein Yang-Mills theory living at the singularity.

Reducing this theory we find that the 7-dim vector potential, $A^a_{\hat{\mu}}$ decomposes into a four dimensional vector, $A^a_\mu$ plus three scalar fields $A^a_m$. The 7-dim scalars, $\phi_{au}$, simply become 4-dim scalars.

$$
\begin{aligned}
b_a{}^m &= -A_{ma}, \\
\rho_a{}^1 &= \sqrt{a^{11}a^{12}}\phi_a{}^3, \\
\rho_a{}^2 &= -\sqrt{a^{21}a^{22}}\phi_a{}^2, \\
\rho_a{}^3 &= \sqrt{a^{31}a^{32}}\phi_a{}^1,
\end{aligned}
$$

A useful redefinition is

$$a^{11} = a^1, \quad a^{12} = a^2, \quad a^{21} = a^3, \quad a^{22} = a^4 \quad a^{31} = a^5, \quad a^{32} = a^6$$

# N=1 Superfields

We split the 4-dimensional field content into “geometric” (or “bulk”) fiel
Which descend from 11-dim SUGRA and “matter fields” which descend
From the 7-dim super Yang-Mills theories at the singularities.

- Geometric

The metric moduli and the 3-form axions combine to form a bosonic superfield

$$T^A = a^A + i\nu^A$$

- Matter

The fields descending from the 7-dim theory at the singularity can be combined to form 4-dim, complex, chiral matter fields

$$C_a{}^m = \rho_a{}^m + i b_a{}^m \, .$$

The reduction of the "bulk" theory (11-dim SUGRA) on a $G_2$ space is well-known and gives rise to the following Kahler potential for the N=1 theory

$$K_0 = -\frac{1}{\kappa_4^2}\sum_{A=0}^{6}\ln\left(T^A+\bar{T}^A\right)+\frac{7}{\kappa_4^2}\ln 2. \qquad \kappa_{11}^2=\kappa_4^2 v_7 \qquad v_7=\int_{\mathcal{Y}}\mathrm{d}^7x$$

Meanwhile, from the 7-dim SU(N) terms we get the following 4-dim Lagrangian terms

$$\begin{aligned}
\mathcal{L}_{4,\mathrm{kin}} &= -\frac{1}{2{\lambda_4}^2}\sqrt{-g}\sum_{m=1}^{3}\left\{\frac{1}{a^{m1}a^{m2}}\left(\mathcal{D}_\mu\rho_a^m\mathcal{D}^\mu\rho^{am}+\mathcal{D}_\mu b_a^m\mathcal{D}^\mu b^{am}\right)\right.\\
&\quad -\frac{1}{3}\sum_{A=0}^{6}\frac{1}{a^{m1}a^{m2}a^A}\partial_\mu a^A\left(\rho_a^m\mathcal{D}^\mu\rho^{am}+b_a^m\mathcal{D}^\mu b^{am}\right)\\
&\quad -\frac{1}{(a^{m1})^2a^{m2}}\rho_a^m\left(\partial_\mu\nu^{m1}\mathcal{D}^\mu b^{am}+\partial_\mu a^{m1}\mathcal{D}^\mu\rho^{am}\right)\\
&\quad \left.-\frac{1}{a^{m1}(a^{m2})^2}\rho_a^m\left(\partial_\mu\nu^{m2}\mathcal{D}^\mu b^{am}+\partial_\mu a^{m2}\mathcal{D}^\mu\rho^{am}\right)\right\},\\
\mathcal{L}_{4,\mathrm{gauge}} &= -\frac{1}{4\lambda_4^2}\sqrt{-g}\left(a^0F_{\mu\nu}^aF_a^{\mu\nu}-\frac{1}{2}\nu^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^aF_{a\rho\sigma}\right),\\
\mathcal{V} &= \frac{1}{4\lambda_4^2a^0}\sqrt{-g}f^a{}_{bc}f_{ade}\sum_{m,n,p=1}^{3}\epsilon_{mnp}\frac{1}{a^{n1}a^{n2}a^{p1}a^{p2}}\left(\rho^{bn}\rho^{dn}\rho^{cp}\rho^{ep}+\rho^{bn}\rho^{dn}b^{cp}b^{ep}\right.\\
&\quad \left.+b^{bn}b^{dn}\rho^{cp}\rho^{ep}+b^{bn}b^{dn}b^{cp}b^{ep}\right)
\end{aligned}$$

# The full N=1 theory in the neighborhood of an isolated singularity

Kahler potential
$$K = \frac{7}{\kappa_4^2}\ln 2 - \frac{1}{\kappa_4^2}\sum_{A=0}^{6}\ln(\tilde{T}^A + \bar{\tilde{T}}^A) + \frac{1}{4\lambda_4^2}\sum_{m=1}^{3}\frac{(C_a^m + \bar{C}_a^m)(C^{am} + \bar{C}^{am})}{(\tilde{T}^{m1} + \bar{\tilde{T}}^{m1})(\tilde{T}^{m2} + \bar{\tilde{T}}^{m2})}$$

Gauge kinetic funtion
$$f_{ab} = \frac{1}{\lambda_4^2}\tilde{T}^0\delta_{ab}\,,$$

Superpotential
$$W = \frac{\kappa_4^2}{24\lambda_4^2}f_{abc}\sum_{m,n,p=1}^{3}\epsilon_{mnp}C^{am}C^{bn}C^{cp}\,.$$

D-terms
$$D_a = \frac{2i\kappa_4^2}{\lambda_4^2}f_{abc}\sum_{m=1}^{3}\frac{C^{bm}\bar{C}^{cm}}{(\tilde{T}^{m1} + \bar{\tilde{T}}^{m1})(\tilde{T}^{m2} + \bar{\tilde{T}}^{m2})}\,.$$

where

$$\tilde{T}^A = T^A - \frac{1}{24\lambda_4^2}\left(T^A + \bar{T}^A\right)\sum_{m=1}^{3}\frac{C_a^m\bar{C}^{am}}{(T^{m1} + \bar{T}^{m1})(T^{m2} + \bar{T}}$$

$$\lambda_{(\tau)}^2 = (4\pi)^{4/3}\frac{v_7^{1/3}}{v_3^{(\tau)}}\kappa_4^{2/3}$$

# Relationship to N=4 super Yang-Mills theory

- This $G_2$ compactification clearly has N=1 SUSY. However, if we neglect the gravity sector (that is, hold constant the geometric moduli, $T^A$), the remaining theory is N=4 SYM, (this makes sense because we are compactifying 7-dim SYM on a 3-torus). We can re-write our results in N=4 language.
- The N=4 SYM Lagrangian

$$\mathcal{L}_{N=4} = -\frac{1}{4g^2}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G_{a\rho\sigma} - \frac{1}{2}\left(\mathcal{D}_\mu A^a_m \mathcal{D}^\mu A^m_a - \frac{1}{2}\mathcal{D}_\mu B^a_m \mathcal{D}^\mu B^m_a\right)$$
$$+ \frac{g^2}{4}\mathrm{tr}\left([A_m, A_n][A^m, A^n] + [B_m, B_n][B^m, B^n] + 2[A_m, B_n][A^m, B^n]\right) .$$

This is exactly our 4-dim effective theory if we define…

$$A^m_a = \frac{1}{\lambda_4\sqrt{a^{m1}a^{m2}}}\rho^m_a \qquad G^a_{\mu\nu} = F^a_{\mu\nu} \qquad g^2 = \frac{\lambda_4^2}{a^0}\,, \qquad \theta = \frac{8\pi^2\nu^0}{\lambda_4^2}$$
$$B^m_a = \frac{1}{\lambda_4\sqrt{a^{m1}a^{m2}}}b^m_a\,;$$

# Interesting N=4 SYM features

- Montonen-Olive and S-duality

If we define $\tau \equiv \frac{\theta}{2\pi} - \frac{4\pi i}{g^2}$ $\tau = -\frac{4\pi i \tilde{T}^0}{\lambda_4^2}$

Then action is invariant under the SL(2, Z) transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

where ad-bc=1, with a,b,c,d ϵ Z. This includes S-duality

$$\tau \to -\frac{1}{\tau}$$

Since the real part of $T^0$ is the volume of the 3-torus, here S-duality is manifested as T-duality!

- "Superconformal" Phase

Unbroken symmetry, in the neighborhood of the singularity.

$$\left[Z^{am}, Z^{bn}\right] = 0 \qquad \langle Z^{am} \rangle = 0$$

- "Coulomb" Phase

Spontaneously broken symmetry, blowing up the singularities

$$\left[Z^{am}, Z^{bn}\right] = 0 \qquad \langle Z^{am} \rangle \neq 0.$$

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- M-theory on $C^2/Z_N$

  1. Inspiration from the structure of 7-d Super Einstein Yang-Mills theory
  2. Constructed a locally SUSY theory to order $h^2$

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- Considering M-theory compactifications, we are lead naturally to G2 spaces
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- "Matching up" to the smooth $G_2$ case.

# Co-dimension 7 singularities

- In order to incorporate charged chiral matter, we must intersect the co-dimension 4 and 7 singularities
- The fixed plane of the co-dim 4 singularity must intersect the tip of the cone (co-dim 7) singularity.
- No compact examples are known of G2 spaces with conical singularities
- How to get them to intersect?

# The goal…

- Be able to write an explict M-theory effective action in the neighborhood of the two intersecting singularities

$$S=\frac{1}{\kappa^2}\int_{M^{11}} dx^{11}\sqrt{-g}\left(R+\ldots\right)+\frac{1}{\lambda^2}\int_{M^{11}}\delta\left(x^4\right)\left(\text{orbifold singularity}\right)$$

$$+\frac{1}{\rho^2}\int_{M^{11}}\delta\left(x^7\right)\left(\text{conical singularity}\right)$$

This work is in progress…

# Further Directions and Applications

- Compactify on other $G_2$ spaces with Compact subspace different from $T^3$. (Local N=1 SUSY?)
- Generalize the procedure for other ADE singularities
- M-theory on K3 with ADE singularities
- Chiral matter – including co-dimension 7 singularities, d=4, N=1 matter fields
- Duality with type IIA and intersecting branes

# The End

# Comparison to the smooth limit

- We find that we can compare this form of the Kahler potential to the case of a smooth $G_2$ manifold where we have “blown-up” the A-type singularities.
- Physically, this corresponds to assigning VEVs to the real parts of the chiral multiplets along D-flat directions.
- Generically, symmetry is broken to $U(1)^{(N-1)}$
- We find unexpectedly that the results agree exactly with those previously found in the smooth limit (up to a choice of embedding the $U(1)^{(N-1)}$ into SU(N) ).
- Potential applications close to (and at) the singularity. Useful for studying wrapped branes and their associated low energy physics.

# Wilson lines and symmetry breaking

$$U_\gamma = P \exp\left(-i \oint_\gamma X_a {A^a}_m dx^m\right)$$

- The first fundamental group of a 3-torus is $Z^3$. This leads to

| Gauge Group | Residual Gauge Groups from Wilson lines |
|---|---|
| $SU_2$ | $U_1$ |
| $SU_3$ | $SU_2 \times U_1$, $U_1^2$ |
| $SU_4$ | $SU_3 \times U_1$, $SU_2 \times U_1^2$, $SU_2^2 \times U_1$, $U_1^3$ |
| $SU_6$ | $SU_5 \times U_1$, $SU_4 \times U_1^2$, $SU_2 \times SU_3 \times U_1^2$, $SU_2^2 \times U_1^3$, $SU_2 \times U_1^4$, $SU_3 \times U_1^3$, $SU_2 \times SU_4 \times U_1$, $SU_2^3 \times U_1^2$, $SU_3^2 \times U_1$, $U_1^5$ |

11-dim View

Compactification and Wilson lines

4-dim View

Turning on VEVs for certain directions of the scalar fields in the potential

# Flux

- We can consider G- and F- flux and find Gukov-type formulas

The effect of G-flux is

$$W = \frac{1}{4}\int_{y}\left(\frac{1}{2}C + i\varphi\right) \wedge Gy$$

Similarly, F-flux

$$W = \frac{\kappa_4^2}{16\lambda_4^2}\frac{1}{v_3}\int_{T^3}\omega_{CS}\,.$$

where

$$\omega_{CS} = \left(\mathcal{F}^a \wedge \mathcal{C}_a - \frac{1}{3}f_{abc}\mathcal{C}^a \wedge \mathcal{C}^b \wedge \mathcal{C}^c\right)$$

$$\mathcal{C}_a = \rho_{am}\mathrm{d}x^m + ib_{am}\mathrm{d}x^m$$

$$\mathcal{F}^a = \mathrm{d}\mathcal{C}^a + f^a{}_{bc}\mathcal{C}^b \wedge \mathcal{C}^c.$$