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Outline

• Introduction
  – M-theory phenomenology
  – $G_2$ Spaces: Why we’re interested. The problems.

• M-theory on $C^2/Z_N$
  – Motivation: Horava-Witten Theory
  – A similar construction on $C^2/Z_N$

• A $G_2$ Compactification with singularities
  – The theory on a $G_2$ orbifold
  – Wilson Lines and Flux
  – Relationship to N=4 Super Yang-Mills theory
  – Blow-ups and the smooth limit

• Conclusion
  – Results and future directions
M-theory
Compactifications

11 Dimensional M-theory

4-dimensional Physics?
M-theory Compactifications

$M_{11} = M_4 \times X_7$

11 Dimensional M-theory

4-dimensional Physics
M-theory Compactifications

$M_{11} = M_4 \times X_7$

Two approaches:
2. $X$ is a manifold with boundary and $\partial X$ is Calabi-Yau
2. $X$ is a more general compact 7-dimensional space

11 Dimensional M-theory

4-dimensional Physics ?
M-theory Compactifications

11 Dimensional M-theory

Compact internal spaces

interesting 4-d phenomenology

\[ M_{11} = M_4 \times X_7 \]

Two approaches:
1. \( X \) is a manifold with boundary and \( \partial X \) is Calabi-Yau
2. \( X \) is a more general compact 7-dimensional space

4-dimensional Physics?
Spinors

- Spinor representations for the 11-d theory decompose as $\text{SO}(1,10) \rightarrow 32 \rightarrow 4+8$
- For $N=1$ SUSY in 4-d we need 1 covariantly constant spinor on $X_7$

It turns out...

A 7-dimensional $G_2$ space comes with exactly such a structure.

Compactification on a $G_2$ space breaks supersymmetry to 1/8 of the original amount ($N=1$ in 4-d).

So, what is a $G_2$ space?...
Recall, the exceptional Lie-group $G_2$ is...

- 14-real dimensional
- The automorphism group of the octonions

- A $G_2$ space is a 7-dim space with $G_2$ holonomy

- $G_2$ holonomy manifold $\rightsquigarrow$ Ricci flat
Properties of $G_2$ spaces

- The space comes equipped with a smooth 3-form, $\varphi_{mnp}$, which is isomorphic to the “flat” 3-form

$$\phi_0 = \left[ d\mathbf{x}^{123} + d\mathbf{x}^{145} + d\mathbf{x}^{167} + d\mathbf{x}^{246} - d\mathbf{x}^{257} - d\mathbf{x}^{347} - d\mathbf{x}^{356} \right]$$
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$$\phi_0 = \left[ dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \right]$$

- This generates an associated metric and a single globally defined spinor

$$g_{mn} = \phi_{mlp} \phi_{nqr} \phi_{stu} \epsilon^{lpqrstu} \quad \phi_{mnp} = i \bar{\eta} \gamma_{mnp} \eta$$

- The pair ($\phi_0$, $g$) together form what is called a “$G_2$ structure”

- The subgroup of GL(7,R) preserving $\phi_0$ is $G_2$
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• Several systematic attempts have been made to construct $G_2$ spaces. Examples include the work of Joyce (orbifolded tori) and Hitchin (non-compact spaces).

• It is hard to make $G_2$ Spaces…
  • No $G_2$ version of Yau’s theorem
  • Generally a complicated object!
M-theory compactification on a $G_2$ manifold

- Witten, Papadopoulos and Townsend compactified M-theory on a smooth 7-manifold with $G_2$ holonomy. They found the following field content in 4-d...

- **Abelian vector multiplets**
  - $b_2(X)$ of them, descending from the 3-form of 11-d SUGRA

- **Uncharged chiral multiplets**
  - $b_3(X)$ of them, descending from the metric moduli of $X$ and the associated axions.

...This is clearly unphysical!

For reasonable 4-dim physics, we need:
- Non-Abelian gauge fields
- Charged chiral matter
The problem gets harder…

In the late 90’s it was found that while smooth $G_2$-spaces are uninteresting, much better things could occur for singular $G_2$-spaces (Witten, Atiyah, Acharya, etc). In the neighborhoods of these singularities it turns out that…

Co-dim 4 Singularities $\rightarrow$ Non-Abelian Gauge Fields

Co-dim 7 (Conical) Singularities $\rightarrow$ Charged Chiral Fermions

So, for realistic physics, we need to construct $G_2$ spaces in which the co-dimension 4 singular locus intersects a conical singularity…this is hard to do!
The current goal of M-theory phenomenology is to produce an effective action for M-theory in the neighborhood of intersecting co-dimension 4 and 7 singularities.

What are the corrections to 11-dimensional supergravity? What are the properties of the effective 4-dimensional theory?

To answer this question, we turn first to co-dimension four singularities...
Co-dimension 4 Singularities

We are lead to the idea of singular spaces in M-theory through dualities with heterotic strings.

M-Theory compactified on K3 = Heterotic string theory compactified on a 3-Torus

In particular, we are interested in compactifying 11-d supergravity on a space with orbifold singularities of the form,

\[ \mathbb{C}^2 / \Gamma_{ADE} \times B_3 \times M_4 \]

Where \( \Gamma_{ADE} \) is an ADE subgroup of SU(2)

To begin, we’ll look at \( \mathbb{C}^2 / \mathbb{Z}_N \) type singularities
$C^2/Z_N$ Orbifolds

$Z_2$ Example:

$Z_N$ symmetry: $(z_1, z_2) \rightarrow (e^{2\pi i/N} z_1, e^{-2\pi i/N} z_2)$

Orbifold singularity at $z_1 = z_2 = 0$

$Z_N$ orbifold $\rightarrow$ SU(N) gauge fields at $(0,0) \times B_3 \times M_4$
New states at a singularity

Example: $\mathbb{C}^2/\mathbb{Z}_2$

We “cut out” the singularity and replace it with an Eguchi-Hanson space, which comes with:

- 2-cycle, $C$, at the origin
- Associated harmonic form $\omega$

Then there is one $U(1)$ vector, $A$, arising from the 3-form, $C = A \wedge \omega$

In the limit that $C$ shrinks to zero, two additional states arise from a membrane wrapping the 2-cycle.
New SU(N) fields - \(C^2/Z_N\)

Blow up the \(C^2/Z_N\) singularities with a chain of \((N-1)\) 2-cycles, (with harmonic 2-forms, \(\omega_i\)) at the origin (i.e. a Gibbons-Hawking space).

• \(U(1)^{N-1}\) gauge fields, \(A^i\) arise from \(C = A^i \wedge \omega_i\)

• Non-Abelian part of SU(N) arises from membranes wrapping the 2-cycles.

Different origin for Abelian and non-Abelian parts Of SU(N)
M-Theory Inspiration: Horava-Witten Theory

Horava and Witten propose that the strong coupling limit of the 10-dim $E_8 \times E_8$ heterotic string is 11-dim M-theory compactified on $R^{1,9} \times S^1/Z_2$

With the gauge fields entering via 10-d vector multiplets propagating only on the boundary of spacetime.

The new states in M-theory appeared in the form of 2 $E_8$ Super YM multiplets, located on the two 10-d fixed planes of the orbifold.
$H = SU(3)_C \times SU(2)_L \times U(1)_Y$

$N_{\text{gen}} = 3$

$Z_2$ Wilson line

$S^1/Z_2$

fixed plane (visible sector)

M fivebrane (hidden sector) ....... M fivebrane (hidden sector)

fixed plane (hidden sector)
This implies something interesting…

There must exist a supersymmetric coupling of 10-d vector multiplets on the orbifold fixed plane to the 11-d supergravity multiplet propagating in the bulk!

Horava and Witten explicitly construct such a theory in the following steps…

• Require 11-d SUGRA to be consistent with orbifolding
• Impose conditions from anomaly cancellation
• Add global $E_8$ multiplets to the orbifold fixed planes
• Apply Noether procedure to get a locally supersymmetric theory.
Brane/Bulk Coupled Theory

Horava-Witten theory proves to be interesting:
Phenomenology, Newton’s constant, Gluino condensation as susy breaking, domain walls, etc.

What about the same approach to M-theory on other orbifolds?
the question:

Can we explicitly write down 11-d SUGRA on the orbifold $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$ coupled to 7-d SU(N) Super-Yang-Mills theory located on the orbifold fixed plane $\mathbb{R}^{1,6} \times \{0\}$?

Further, can such a construction be directly used in a $G_2$ compactification? That is, can we find the structure of low-energy M-theory near a $\mathbb{C}^2/\mathbb{Z}_N$ singularity embedded into a $G_2$ space?
Overview of the singular M-theory construction:

Constrain 11-d SUGRA to be consistent with orbifolding
Overview of the singular M-theory construction:

- Constrain 11-d SUGRA to be consistent with orbifolding

Truncate 11-d SUGRA on \( C^2/Z_N \)

- 7-d SUGRA + YM with Gauge group \( U(1)^n \) \( n=1,3 \)
Overview of the singular M-theory construction:

Constrain 11-d SUGRA to be consistent with orbifolding

Truncate 11-d SUGRA
On $\mathbb{C}^2/Z_N$

Add SU(N) multiplets

7-d SUGRA+YM
With Gauge group $U(1)^n$, $n=1,3$

7-d SUGRA+YM
With Gauge group $SU(N) \times U(1)^n$
Overview of the singular M-theory construction:

Constrain 11-d SUGRA to be consistent with orbifolding

11-d SUGRA on $\mathbb{C}^2/\mathbb{Z}_N$ coupled to 7-d SU(N) SYM

Truncate 11-d SUGRA on $\mathbb{C}^2/\mathbb{Z}_N$

"lift up" to 11-d

7-d SUGRA+YM With Gauge group $U(1)^n \quad n=1,3$

Add SU(N) multiplets

7-d SUGRA+YM With Gauge group $SU(N) \times U(1)^n$
Review of Einstein Yang-Mills Theory in 7-d (what we’re aiming for...)

- Field content: $(g_{\mu \nu}, C_{\mu \nu \rho}, A_{\mu j}, \sigma, \psi_{\mu}^{i}, \chi^{i})$
- Gravity multiplet
- Gauge multiplet $(A_{\mu}^{a}, \phi^{aij}, \lambda^{ai})$

- R-symmetry index, $i=1,2$
- Gauge group $G$, $a=1,\ldots M$
- The scalars, $\phi^{aij}$ parametrize the coset $SO(3,M)/SO(3) \times SO(M)$
- Symplectic Majorana Spinors
11-d Supergravity on $C^2/Z_N$

$$S_{11} = \frac{1}{\kappa^2} \int_{\mathcal{M}_1^{11}} d^{11}x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} \bar{\Psi}_M \Gamma^{MNP} \nabla_N \Psi_P - \frac{1}{96} G_{MNPQ} G^{MNPQ} ight. \\
- \frac{1}{192} \left( \bar{\Psi}_M \Gamma^{MNPQRS} \Psi_S + 12 \bar{\Psi}_N \Gamma^{PQ} \Psi_R \right) G_{NPQR} \right) \\
- \frac{1}{12\kappa^2} \int_{\mathcal{M}_1^{11}} C \wedge G \wedge G + \ldots$$

Has field content: $(g_{MN}, \Psi_M, C_{MNP})$

$G = dC$

7+4 Coord split: $x^M = (x^\mu, y^A) = (x^\mu, z^p, \bar{z}^{\bar{p}})$

Spinor decomposition: $\Psi = \psi_i(x, y) \otimes \rho^i + \psi_{\bar{j}}(x, y) \otimes \rho^{\bar{j}}$
The Orbifold action

\[ e_{\mu}^{\nu}(x, Ry) = e_{\mu}^{\nu}(x, y), \]
\[ e_{A}^{\nu}(x, Ry) = (R^{-1})_{A}^{B}e_{B}^{\nu}(x, y), \]
\[ e_{\mu}^{A}(x, Ry) = T_{B}^{A}e_{\mu}^{B}(x, y), \]
\[ e_{A}^{B}(x, Ry) = (R^{-1})_{A}^{C}T_{D}^{B}e_{C}^{D}(x, y), \]

\[ C_{\mu\nu\rho}(x, Ry) = C_{\mu\nu\rho}(x, y), \]
\[ C_{\mu\nu\varphi}(x, Ry) = (R^{-1})_{A}^{B}C_{\mu\nu\varphi}(x, y), \text{ etc.} \]
\[ \Psi_{\mu i}(x, Ry) = \Psi_{\mu i}(x, y), \]
\[ \Psi_{\mu\bar{\nu}}(x, Ry) = S_{\bar{i}}^{\bar{j}}\Psi_{\mu\bar{j}}(x, y), \]
\[ \Psi_{A i}(x, Ry) = (R^{-1})_{A}^{B}\Psi_{B i}(x, y), \]
\[ \Psi_{A\bar{i}}(x, Ry) = (R^{-1})_{A}^{B}S_{\bar{i}}^{\bar{j}}\Psi_{B\bar{j}}(x, y). \]

Where

\[ (R^{p}_{q}) = e^{2i\pi/N}1_{2} \quad (R^{\bar{p}}_{q}) = (R^{p}_{\bar{q}}) = 0 \quad (T^{p}_{q}) = e^{2i\pi/N}1_{2} \]
\[ (R^{\bar{p}}_{\bar{q}}) = e^{-2i\pi/N}1_{2} \quad (S_{i}^{j}) = e^{2i\pi/N}1_{2} \quad (T^{\bar{p}}_{\bar{q}}) = e^{-2i\pi/N}1_{2} \]
Different 7-d field content for the $N=2$ and $N>2$ cases

For $N>2$  
\[ g_{\mu\nu}, e^q_p, C_{\mu\nu\rho}, C_{\mu p\bar{q}}, \Psi_{\mu i}, (\Gamma^p \Psi_p)_i \text{ and } (\Gamma^{\bar{p}} \Psi_{\bar{p}})_i \]

For $N=2$  
\[ g_{\mu\nu}, e^B_A, C_{\mu\nu\rho}, C_{\mu AB}, \Psi_{\mu i} \text{ and } \Psi_{A\bar{i}} \]

The two cases lead to different structure
Field Identifications

For $\mathbb{Z}_N$

$$\sigma = \frac{3}{20} \ln \det g_{AB},$$

$$\tilde{g}_{\mu\nu} = e^{\frac{4}{3}\sigma} g_{\mu\nu},$$

$$\psi_{\mu i} = \Psi_{\mu i} e^{\frac{1}{3}\sigma} - \frac{1}{5} \gamma^{\mu} \left( \Gamma^{A} \Psi_{A} \right)_i e^{-\frac{1}{3}\sigma},$$

$$\tilde{C}_{\mu\nu\rho} = C_{\mu\nu\rho},$$

$$\chi_i = \frac{3}{2\sqrt{5}} \left( \Gamma^{A} \Psi_{A} \right)_i e^{-\frac{1}{3}\sigma},$$

$$F_{\mu\nu}^{I} = -\frac{i}{2} \text{tr} \left( \sigma^{I} G_{\mu\nu} \right),$$

$$\lambda_i = \frac{i}{2} \left( \Gamma^{p} \Psi_{p} - \Gamma^{\bar{p}} \Psi_{\bar{p}} \right)_i e^{-\frac{1}{3}\sigma},$$

$$\ell_{I}^{J} = \frac{1}{2} \text{tr} \left( \bar{\sigma}_{I} v \sigma^{J} v^{\dagger} \right).$$

$$G_{\mu\nu} \equiv (G_{\mu\nu p\bar{q}}), \quad v \equiv \left( e^{5\sigma/6} e^{\bar{p}/\bar{q}} \right).$$

And for $\mathbb{Z}_2$

$$F_{\mu\nu}^{I} = -\frac{1}{4} \text{tr} \left( T^{I} G_{\mu\nu} \right),$$

$$\ell_{I}^{J} = \frac{1}{4} \text{tr} \left( \bar{T}_{I} v T^{J} v^{T} \right).$$
Reviewing the pieces…
The ‘component’ Lagrangians

• Begin with…
  \[ L_{11} \]

• Truncate under orbifold action to…
  \[ L_7^{(n)} \]

• Add SU(N) Yang-Mills
  \[ L_{SU(N)} \]

• Complete theory is M-theory on \( \mathbb{C}^2/\mathbb{Z}_N \)
  \[ L_{11} + \delta^{(4)}(L_{SU(N)} - L_7^{(n)}) \]
\[ \mathcal{L}_{7}^{(n)} = \frac{1}{\kappa_{7}^2} \sqrt{-\tilde{g}} \left\{ \frac{1}{2} R - \frac{1}{2} \bar{\psi}_{\mu} \gamma^{\mu\nu\rho} \hat{D}_{\nu} \psi_{\rho i} - \frac{1}{4} e^{-2\sigma} \left( \ell_{I} \ell_{J} + \ell_{I} \ell_{J} \right) F_{\mu\nu}^{I} F^{J\mu\nu} \right. \\
- \frac{1}{96} e^{4\sigma} \tilde{G}_{\mu\nu\rho\sigma} \tilde{G}^{\mu\nu\rho\sigma} - \frac{1}{2} \bar{\chi} \gamma^{\mu} \hat{D}_{\mu} \chi_{i} - \frac{5}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma + \frac{\sqrt{5}}{2} \left( \bar{\chi} \gamma^{\mu} \psi_{\mu i} + \bar{\psi} \gamma^{\mu} \psi_{\mu j} \right) \partial_{\nu} \sigma \\
- \frac{1}{2} \bar{\chi}^{\alpha i} \gamma^{\mu} \hat{D}_{\mu} \lambda_{\alpha i} - \frac{1}{2} \mu_{\alpha j} \mu^{\alpha j} - \frac{1}{\sqrt{2}} \left( \bar{\chi}^{\alpha i} \gamma^{\mu} \psi_{\mu j} + \bar{\lambda}^{\alpha i} \psi_{\mu j} \right) \mu_{\mu j} \right\} \\
+ e^{2\sigma} \tilde{G}_{\mu\nu\rho\sigma} \left[ \frac{1}{192} \left( 12 \bar{\psi}_{\mu i} \gamma^{\nu\rho\sigma} \psi_{\nu i} + \bar{\psi}_{\chi} \gamma^{\mu\nu\rho\sigma} \psi_{\chi i} \right) + \frac{1}{48 \sqrt{5}} \left( 4 \bar{\chi} \gamma^{\mu\nu\rho} \psi_{\nu i} \right) \\
\left. - \bar{\chi} \gamma^{\mu\nu\rho\sigma} \psi_{\chi i} \right) - \frac{1}{320} \bar{\chi} \gamma^{\mu\nu\rho\sigma} \chi_{i} + \frac{1}{192} \bar{\lambda}^{\alpha i} \gamma^{\mu\nu\rho\sigma} \lambda_{\alpha i} \right\] \\
- i e^{-\sigma} F_{\mu\nu}^{I} \ell_{I} \ell_{j} \left[ \frac{1}{4 \sqrt{2}} \left( \bar{\psi}_{\nu} \gamma^{\mu\rho\sigma} \psi_{\sigma j} + 2 \bar{\psi}_{\mu i} \psi_{\nu} \right) + \frac{1}{2 \sqrt{10}} \left( \bar{\chi} \gamma^{\mu\rho\sigma} \psi_{\rho j} - 2 \bar{\chi} \gamma^{\mu\rho} \psi_{\rho j} \right) \\
+ \frac{3}{20 \sqrt{2}} \bar{\chi} \gamma^{\mu} \chi_{j} - \frac{1}{4 \sqrt{2}} \bar{\lambda}^{\alpha i} \gamma^{\mu\nu} \lambda_{\alpha j} \right] \\
+ e^{-\sigma} F_{\mu\nu}^{I} \ell_{I} \ell_{\alpha} \left[ \frac{1}{4} \left( 2 \bar{\lambda}^{\alpha i} \gamma^{\mu} \psi_{\nu i} - \bar{\lambda}^{\alpha i} \gamma^{\mu\rho\sigma} \psi_{\rho i} \right) + \frac{1}{2 \sqrt{5}} \bar{\lambda}^{\alpha i} \gamma^{\mu\nu} \chi_{i} \right] \\
- \frac{1}{96} e^{\mu\nu\rho\sigma\kappa\lambda} C_{\mu\nu\rho}^{I} F_{\sigma\kappa}^{I} F_{I\lambda\tau} \right\} .
\]
To the bulk 11-d theory we now want to add SU(N) multiplets on the orbifold fixed plane. So the coset structure of the scalar fields of the 7-d Einstein Yang-Mills theory will become

$$\text{SO}(3, n+N^2-1)/\text{SO}(3) \times \text{SO}(n+ N^2 -1)$$

Where \( n=1,3 \).

Note that the gravity and SU(N) scalars have now become entangled.
The Action Schematically

\[ S_{11-7} = \int_{M_{11}^N} d^{11}x \left[ \mathcal{L}_{11} + \delta^{(4)}(y^A)\mathcal{L}_{\text{brane}} \right], \]

\[ \mathcal{L}_{\text{brane}} = \mathcal{L}_{\text{SU(N)}} - \mathcal{L}_{7}^{(n)}. \]

Susy transformations

\[ \delta_{11} = \delta_{11}^{11} + \kappa^{8/9} \delta^{(4)}(y^A)\delta_{11}^{\text{brane}} \]
\[ \delta_{7} = \delta_{7}^{\text{SU(N)}} \]

Where \[ \delta_{11}^{\text{brane}} = \delta_{11}^{\text{SU(N)}} - \delta_{11}^{11} \]

\( \delta_{11} \) - acts on bulk fields \hspace{0.5cm} \delta_{11}^{11} - 11-d susy transformations
\( \delta_{7} \) - acts on brane fields \hspace{0.5cm} \delta_{7} - 7-d susy transformations
Expansion in coupling constants:

Because the coset structure of the 7-d theory entangles the gravity and scalar fields in the coset,

\[ \text{SO}(3, n+N^2-1)/\text{SO}(3) \times \text{SO}(n+N^2-1), \]

in order to quantitatively understand the corrections, we perform an expansion in the coupling constants:

\[ \mathcal{L}_{SU(N)} = \kappa_7^{-2} \left( \mathcal{L}_{(0)} + h^2 \mathcal{L}_{(2)} + h^4 \mathcal{L}_{(4)} + \ldots \right) \]

Where

\[ \kappa_7 = \kappa^{5/9}, \quad h = \kappa_7/g_{YM} \]

and

\[ \delta^{SU(N)}_{11} = \delta_{11}^{(0)} + h^2 \delta_{11}^{(2)} + h^4 \delta_{11}^{(4)} + \ldots \]
We expand to order $\hbar^2$

The coset representative then takes the form

$$L = \begin{pmatrix} \ell + \frac{1}{2} \hbar^2 \ell \Phi^T \Phi & m & h \ell \Phi^T \\ h \Phi & 0 & 1_{N^2-1} + \frac{1}{2} \hbar^2 \Phi \Phi^T \end{pmatrix}$$

Unlike Horava-Witten, the 7-d theory does not fix the value of $g_{YM}$. However from comparing with IIA D6 branes, (Friedmann and Witten) one finds,

$$g_{YM}^2 = (4\pi)^{4/3} \kappa^{2/3}$$

• The coupled theory is supersymmetric to order $\hbar^2$
• At order $\hbar^4$ we encounter $\delta(0)$ singularity (also present in Horava-Witten theory).
\[ L_{\text{brane}} = \frac{1}{g_{\text{YM}}^2} \sqrt{-\tilde{g}} \left\{ \frac{1}{4} e^{-2\sigma F_{\mu \nu}^a F_{\mu \nu}^a} - \frac{1}{2} \tilde{D}_\mu \phi_i \tilde{D}_\mu \phi^{aj} - \frac{1}{2} \bar{\lambda}^{ai} \gamma^{\mu \nu} \tilde{D}_\mu \lambda_{ai} - e^{-2\sigma} \ell_i^j \phi^j \ell^i F_{\mu \nu} F^{a \mu \nu} \right. \\
- \frac{1}{2} e^{-2\sigma} \ell_i^j \phi^j \ell^j l^k \phi_{k a} F_{\mu \nu} F^{j \mu \nu} - \frac{1}{2} p_{\mu \alpha} j \phi^j l \phi^{\mu \alpha k} l \phi_{k l} \\
+ \frac{1}{4} \phi_{a k} \tilde{D}_\mu \phi^{ak} \left( \bar{\psi}_j \gamma^{\mu \nu} \psi_{j i} + \bar{\lambda}^{aj} \gamma^{\mu \nu} \lambda_{ai} \right) \\
- \frac{1}{2\sqrt{2}} \left( \bar{\lambda}^{ai} \gamma^{\mu \nu} \psi_{j i} + \bar{\lambda}^{ai} \gamma^{\nu \mu} \psi_{j i} \right) \tilde{D}_\nu \phi_{a j} \\
+ \frac{1}{192} e^{2\sigma} \tilde{G}_{\mu \nu \rho \sigma} \bar{\lambda}^{ai} \gamma^{\mu \nu \rho \sigma} \lambda_{ai} + \frac{i}{4\sqrt{2}} e^{-\sigma} F_{\mu \nu} \ell_i^j \bar{\lambda}^{ai} \gamma^{\mu \nu} \lambda_{aj} \\
- \frac{i}{2} e^{-\sigma} \left( F_{\mu \nu} \ell_i^j k \phi_{k a} \phi_{j i}^a + 2F_{\mu \nu}^a \phi_{j i}^a \right) \left[ \frac{1}{4\sqrt{2}} \left( \bar{\psi}_\rho \gamma^{\mu \nu \rho \sigma} \psi_\sigma j + 2 \bar{\psi}_\mu i \psi_\nu j \right) \\
+ \frac{3}{20\sqrt{2}} \bar{\chi}^{ai} \gamma^{\mu \nu} \chi_j - \frac{1}{4\sqrt{2}} \bar{\lambda}^{ai} \gamma^{\mu \nu} \lambda_{aj} + \frac{1}{2\sqrt{10}} \left( \bar{\chi}^{ai} \gamma^{\mu \nu} \psi_{j i} - 2 \bar{\chi}^{ai} \gamma^{\mu \nu} \psi_{j i} \right) \right] \\
+ e^{-\sigma} F_{a \mu \nu} \left[ \frac{1}{4} \left( 2 \bar{\lambda}^{ai} \gamma^{\mu \nu} \psi_{j i} - \bar{\lambda}^{ai} \gamma^{\mu \nu \rho} \psi_{j i} \right) + \frac{1}{2\sqrt{5}} \bar{\lambda}^{ai} \gamma^{\mu \nu} \chi_i \right] \\
+ \frac{1}{4} e^{2\sigma} f_{a b c} f_{def} \phi_{k \phi}^{b i} \phi_{j \phi}^{d j} \phi_{l \phi}^{e l} - \frac{1}{2} e^{2\sigma} f_{a b c} \phi_{k \phi}^{b i} \phi_{j \phi}^{d j} \phi_{l \phi}^{e l} \left( \bar{\psi}_\mu \gamma^{\mu \nu} \lambda_{ai} + \frac{2}{\sqrt{5}} \bar{\psi}_\mu \gamma^{\mu \nu} \lambda_{ai} \right) \\
- \frac{i}{\sqrt{2}} e^{2\sigma} f_{a b c} \phi_{j \phi}^{d j} \phi_{c \phi}^{l \phi} \phi_{b \phi}^{i} \phi_{i \phi}^{l \phi} \phi_{j \phi}^{d j} \phi_{l \phi}^{e l} \left( 5 \bar{\psi}_\mu \gamma^{\mu \nu} \psi_{j i} + 2 \sqrt{5} \bar{\psi}_\mu \gamma^{\mu \nu} \chi_i \right) \\
+ 3 \bar{\chi}^{ai} \chi_i - 5 \bar{\lambda}^{ai} \lambda_{ai} \right\}.
\[ L_{\text{brane}} = \frac{1}{g^2_{\text{YM}}} \sqrt{-g} \left\{ -\frac{1}{4} e^{-2\sigma} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2} \partial_\mu \phi_a \partial_\nu \phi^a_i - \frac{1}{2} \bar{\lambda}^{ai} \gamma^\mu \partial_\mu \lambda_{ai} - e^{-2\sigma} \ell_I \phi_i^j \gamma^\mu \gamma_F^I \phi_i^j \gamma^\mu \phi_i^j \right\} \\
- \frac{1}{2} e^{-2\sigma} \frac{\ell_I}{j} \phi^{aj} \ell_j \bar{\phi}_{a k} F_{\mu\nu}^I F_{\mu\nu}^k - \frac{1}{2} p_{\mu\nu}^i \phi_j^a i \partial_\gamma \phi^{alk} \partial_\mu \phi_i^{al} \\
+ \frac{1}{4} \phi_{a k} \bar{\phi}_k \gamma^{al} \left( \bar{\psi}_j \gamma^{\mu\nu} \psi_{ij} + \bar{\lambda}^{aj} \gamma^\mu \lambda_{ai} \right) \\
- \frac{1}{2 \sqrt{2}} \left( \bar{\lambda}^{ai} \gamma^\mu \psi_{ij} + \bar{\lambda}^{ai} \gamma^\nu \psi_{ij} \right) \partial_\nu \phi_i^j \right\} \\
+ \frac{1}{192} e^{2\sigma} \gamma^{\mu\nu\rho\sigma} \bar{\lambda}^{ai} \gamma_{\mu\nu\rho\sigma} \lambda_{ai} + \frac{i}{4 \sqrt{2}} e^{-\sigma} F_{\mu\nu}^I \ell_I \bar{\phi}_a^i \gamma^\mu \gamma^\nu \lambda_{ai} \\
- \frac{i}{2} e^{-\sigma} \left( F_{\mu\nu}^I \ell_I \bar{\phi}_a^i + 2 F_{\mu\nu}^a \phi_i^j \right) \left[ \frac{1}{4 \sqrt{2}} \left( \bar{\psi}_j^e \gamma^{\mu\nu\rho\sigma} \psi_{ij} + 2 \bar{\psi}_j^{\mu\nu} \psi_{ij} \right) \\
+ \frac{3}{20 \sqrt{2}} (\bar{\psi}_j^e \gamma^{\mu\nu} \psi_{ij} - \frac{1}{4 \sqrt{2}} \bar{\psi}_j^e \gamma^{\mu\nu} \lambda_{ai}) \lambda_{aj} + \frac{1}{2 \sqrt{10}} (\bar{\psi}_j^e \gamma^{\mu\nu} \psi_{ij} - \bar{\psi}_j^e \gamma^{\mu\nu} \psi_{ij}) \right] \\
+ e^{-\sigma} F_{\mu\nu}^a \left[ \frac{1}{4} \left( 2 \bar{\psi}_j^e \gamma^{\mu\nu} \psi_{ij} - \bar{\psi}_j^e \gamma^{\mu\nu} \psi_{ij} \right) + \frac{1}{2 \sqrt{5}} \gamma^{\mu\nu} \phi_i^j \right] \\
+ \frac{1}{4} e^{2\sigma} f_{bc}^a f_{dea} \bar{\phi}_a^i \gamma^{\mu\nu} \psi_{ij} + \frac{1}{2} e^{2\sigma} f_{abc} \phi_i^j \phi^a_i \gamma^{\mu\nu} \phi_i^j \phi^a_i + \frac{1}{2} e^{2\sigma} f_{abc} \phi_i^j \phi^a_i \gamma^{\mu\nu} \phi_i^j \phi^a_i \\
- \frac{i}{\sqrt{2}} e^{\sigma} f_{abc} \phi_i^j \gamma^{\mu\nu} \psi_{ij} + \frac{i}{60 \sqrt{2}} e^{\sigma} f_{abc} \phi_i^j \gamma^{\mu\nu} \psi_{ij} + \frac{1}{2} e^{\sigma} f_{abc} \phi_i^j \gamma^{\mu\nu} \psi_{ij} + \frac{1}{2} e^{\sigma} f_{abc} \phi_i^j \gamma^{\mu\nu} \psi_{ij} \\
+ \frac{1}{96} e^{\mu\nu\rho\sigma\lambda\tau} \partial_\mu \phi_i^j \gamma_{\mu\nu} \psi_{ij} + \frac{1}{96} e^{\mu\nu\rho\sigma\lambda\tau} \partial_\mu \phi_i^j \gamma_{\mu\nu} \psi_{ij} + \frac{1}{96} e^{\mu\nu\rho\sigma\lambda\tau} \partial_\mu \phi_i^j \gamma_{\mu\nu} \psi_{ij} + \frac{1}{96} e^{\mu\nu\rho\sigma\lambda\tau} \partial_\mu \phi_i^j \gamma_{\mu\nu} \psi_{ij} \right\} 
\]
Supersymmetry corrections

\[ \delta^{\text{brane}} \psi_{\mu i} = \frac{k_7^2}{g_{\text{YM}}^2} \left\{ \frac{1}{2} \left( \phi_{ak} j D_\mu \phi_{i k}^a j - \phi_{ai} j D_\mu \phi_{ak}^a j \right) \varepsilon_j - \frac{i}{15\sqrt{2}} \gamma_{\mu \nu} \varepsilon_{fi} j f_{ab} c_{\phi_{ik}}^a \phi_{k l}^b \phi_{j l}^c \gamma^\nu \varepsilon_{i j} \right\} + \frac{i}{10\sqrt{2}} \left( \gamma_{\mu \nu} - 8 \delta_{\mu \nu} \gamma^\rho \right) \varepsilon_{i j} \left( F_{\nu \rho}^{I} l_{\mu}^k \phi_{a i}^a l_{\phi_{a j}^a}^k + 2 F_{\nu \rho}^{a} \phi_{a i}^a \right) e^{-\sigma} \]

\[ \delta^{\text{brane}} \chi_i = \frac{k_7^2}{g_{\text{YM}}^2} \left\{ -\frac{i}{2\sqrt{10}} \gamma^{\mu \nu} \varepsilon_{i j} \left( F_{\mu \nu}^{I} l_{\phi_{a i}^a}^k + \phi_{a i}^a \right) e^{-\sigma} \right\} + \frac{i}{3\sqrt{10}} \varepsilon_{i f} a_{\phi_{a i}^a} b_{\phi_{a i}^a} j \phi_{j l}^c \gamma^\nu \varepsilon_{i j} \right\} \]

\[ \ell_i^I \delta^{\text{brane}} A_{\mu}^I = \frac{k_7^2}{g_{\text{YM}}^2} \left\{ \left( \frac{i}{\sqrt{2}} \bar{\psi}_{\mu}^k \varepsilon_{i l} - \frac{i}{\sqrt{10}} \bar{\chi}_{\mu} \varepsilon_{i l} \right) \phi_{i j}^a \phi_{a i}^a \gamma^\nu \varepsilon_{i j} - \bar{\varepsilon}^k \gamma_{\mu} \lambda_{k}^a \phi_{a i}^a \gamma^\nu \varepsilon_{i j} \right\} \]

\[ \ell_i^I \delta^{\text{brane}} A_{\mu}^I = 0, \]

\[ \delta^{\text{brane}} \ell_i^I = \frac{k_7^2}{g_{\text{YM}}^2} \left\{ \frac{i}{\sqrt{2}} \left[ \bar{\varepsilon}^k \lambda_{a l} \phi_{a j}^i \phi_{a i}^a \ell_{a i}^1 + \bar{\varepsilon}^i \lambda_{a k} \phi_{a j}^i \ell_{a i}^1 \right] - \left( \bar{\varepsilon}^i \lambda_{a j} - \frac{1}{2} \delta_j^l \varepsilon^{m} \gamma_{a m} \right) \phi_{a k}^i \ell_{a i}^1 \right\} \]

\[ \delta^{\text{brane}} \ell_i^I = \frac{k_7^2}{g_{\text{YM}}^2} \left\{ \frac{i}{\sqrt{2}} \bar{\varepsilon}^i \lambda_{a j}^I \phi_{a i}^a \phi_{a i}^a \ell_{a i}^1 \right\}, \]

\[ \delta^{\text{brane}} \lambda_i^\alpha = \frac{k_7^2}{g_{\text{YM}}^2} \left\{ \frac{i}{\sqrt{2}} \gamma_{\mu} \varepsilon_{j a} \phi_{a i}^a \phi_{a i}^a \right\}, \]
The Brane Bosonic Theory

\[ S_{7,\text{bos}} = \frac{1}{g_{\text{YM}}^2} \int_{y=0} d^7 x \sqrt{-g} \left( -\frac{1}{4} H_{a b} F_{\mu \nu}^a F^{b \mu \nu} - \frac{1}{2} H_{a I} F_{\mu \nu}^a F^{I \mu \nu} - \frac{1}{4} (\delta H)_{I J} F_{\mu \nu}^I F^{I \mu \nu} \right. \]
\[ \left. \left. -\frac{1}{2} e^\tau \hat{D}_\mu \phi^i_a \hat{D}^\mu \phi^{a j}_i - \frac{1}{2} (\delta K)^{\alpha j}_{i k} p_{\mu \alpha}^i j p^\mu_{\beta k} + \frac{1}{4} D_{a i} D_{a j} \right) \right) \]
\[ - \frac{1}{4 g_{\text{YM}}^2} \int_{y=0} C \wedge F^a \wedge F_a, \]  

(5)

Gauge-Kinetic Function (SU(N))

\[ H_{a b} = \delta_{a b}, \]

\[ H_{a I} = 2 \ell^i_{I j} \phi^i_{a j}, \]

\[ (\delta H)_{I J} = 2 \ell^i_{I j} \phi^i_{a j} \ell^k_{J l} \phi^l_{a k}, \]

\[ (\delta K)^{\alpha j}_{i k} = e^\tau \delta^{\alpha \beta}_{i} \phi^\beta_{a j} \phi^{a l}_{i k}, \]

\[ D^{a i}_{j} = e^\tau f_{a b c}^i \phi^b_{j} \phi^{c k}_{j}. \]

Gauge-Kinetic Function for gravi-photons

Contributes to D-term potential

Where we recall, the bulk fields are coupled in this action through

\[ F_{\mu \nu}^I = -\frac{i}{2} \text{tr} \left( \sigma^I G_{\mu \nu} \right), \]

\[ \ell^i_{I j} = \frac{1}{\sqrt{2}} \ell^u_I (\sigma_u)^i j, \]

\[ \tau = \frac{1}{2} \ln \det g_{A B}, \]

\[ p_{\mu \alpha}^i j = \ell^I_{\alpha} \delta_\mu \ell^i_{I j}, \]

\[ D^a = \frac{1}{2} e^\tau f_{b c}^a [\phi^b, \phi^c], \]

\[ V = \frac{1}{4 g_{\text{YM}}^2} \text{tr} (D^a D_a). \]
After constructing the 7-dimensional theory, we are now ready to embed our singular neighborhood into a $G_2$ space…
G₂ Compactification

After constructing the 7-dimensional theory, we are now ready to embed our singular neighborhood into a G₂ space…

• We will utilize G₂ orbifolds, $T^7/\Gamma$, constructed by dividing a 7-torus, $T^7$, by a discrete symmetry, $\Gamma$, such that the resulting singularities are of co-dimension 4 and A-type. We choose particular symmetry groups such that the singular loci will always be 3-tori, $T^3$. 
After constructing the 7-dimensional theory, we are now ready to embed our singular neighborhood into a $G_2$ space…

- We will utilize $G_2$ orbifolds, $T^7/\Gamma$, constructed by dividing a 7-torus, $T^7$, by a discrete symmetry, $\Gamma$, such that the resulting singularities are of co-dimension 4 and A-type. We choose particular symmetry groups such that the singular loci will always be 3-tori, $T^3$.
- In the neighborhood of a singularity, the $G_2$ space looks like $\mathbb{C}^2/\mathbb{Z}_N \times T^3$. 
After constructing the 7-dimensional theory, we are now ready to embed our singular neighborhood into a $G_2$ space...

- We will utilize $G_2$ orbifolds, $T^7/\Gamma$, constructed by dividing a 7-torus, $T^7$, by a discrete symmetry, $\Gamma$, such that the resulting singularities are of codimension 4 and A-type. We choose particular symmetry groups such that the singular loci will always be 3-tori, $T^3$.
- In the neighborhood of a singularity, the $G_2$ space looks like $C^2/Z_N \times T^3$.
- While the full 4-dimensional theory will be $N=1$ supersymmetric, the gauge sub-sectors associated to each singularity have enhanced $N=4$ supersymmetry.
To begin the compactification, we write the 11-dimensional metric as

\[ ds^2 = \left( \prod_{A=1}^{7} R^A \right)^{-1} g_{\mu\nu} dx^\mu dx^\nu + \sum_{A=1}^{7} (R^A dx^A)^2 \]

where the $R^A$ are the seven radii of the $T^7$

There exists a $G_2$ structure, a harmonic 3 form associated to the metric above

\[ \varphi = R^1 R^2 R^3 dx^1 \wedge dx^2 \wedge dx^3 + R^1 R^4 R^5 dx^1 \wedge dx^4 \wedge dx^5 - R^1 R^6 R^7 dx^1 \wedge dx^6 \wedge dx^7 \\
+ R^2 R^4 R^6 dx^2 \wedge dx^4 \wedge dx^6 + R^2 R^5 R^7 dx^2 \wedge dx^5 \wedge dx^7 + R^3 R^4 R^7 dx^3 \wedge dx^4 \wedge dx^7 \\
- R^3 R^5 R^6 dx^3 \wedge dx^5 \wedge dx^6. \]

where the some of the $R^A$ are related by orbifolding. From this we define the metric moduli

\[ a^0 = R^1 R^2 R^3, \quad a^1 = R^1 R^4 R^5, \quad a^2 = R^1 R^6 R^7, \quad a^3 = R^2 R^4 R^6, \]
\[ a^4 = R^2 R^5 R^7, \quad a^5 = R^3 R^4 R^7, \quad a^6 = R^3 R^5 R^6. \]

Similarly, the 3-form of 11-dim SUGRA can be expanded as

\[ C = \nu^0 dx^1 \wedge dx^2 \wedge dx^3 + \nu^1 dx^1 \wedge dx^4 \wedge dx^5 - \nu^2 dx^1 \wedge dx^6 \wedge dx^7 + \nu^3 dx^2 \wedge dx^4 \wedge dx^6 \\
+ \nu^4 dx^2 \wedge dx^5 \wedge dx^7 + \nu^5 dx^3 \wedge dx^4 \wedge dx^7 - \nu^6 dx^3 \wedge dx^5 \wedge dx^6. \]
Field Content from the singularity

In addition to the field content from 11-dim SUGRA, we also have contributions from the 7-dim Einstein Yang-Mills theory living at the singularity.

Reducing this theory we find that the 7-dim vector potential, $A_\mu^a$, decomposes into a four dimensional vector, $A_\mu^a$ plus three scalar fields $A_m^a$. The 7-dim scalars, $\phi_{au}$, simply become 4-dim scalars.

\[
\begin{align*}
 b_a^m & = -A_{ma}, \\
 \rho_a^1 & = \sqrt{a^{11}a^{12}}\phi_a^3, \\
 \rho_a^2 & = -\sqrt{a^{21}a^{22}}\phi_a^2, \\
 \rho_a^3 & = \sqrt{a^{31}a^{32}}\phi_a^1,
\end{align*}
\]

A useful redefinition is

\[
\begin{align*}
 a^{11} & = a^1, & a^{12} & = a^2, & a^{21} & = a^3, & a^{22} & = a^4, & a^{31} & = a^5, & a^{32} & = a^6
\end{align*}
\]
N=1 Superfields

We split the 4-dimensional field content into “geometric” (or “bulk”) fields which descend from 11-dim SUGRA and “matter fields” which descend from the 7-dim super Yang-Mills theories at the singularities.

• Geometric

The metric moduli and the 3-form axions combine to form a bosonic superfield

\[ T^A = a^A + i\nu^A \]

• Matter

The fields descending from the 7-dim theory at the singularity can be combined to form 4-dim, complex, chiral matter fields

\[ C_a^m = \rho_a^m + i b_a^m \]
The reduction of the “bulk” theory (11-dim SUGRA) on a $G_2$ space is well-known and gives rise to the following Kahler potential for the $N=1$ theory

$$K_0 = -\frac{1}{\kappa_4^2} \sum_{A=0}^{6} \ln \left(T^A + \bar{T}^A\right) + \frac{7}{\kappa_4^2} \ln 2.$$  \hspace{1cm} \kappa_{11}^2 = \kappa_4^2 v_7 \hspace{1cm} v_7 = \int_Y d^7 x$$

Meanwhile, from the 7-dim $SU(N)$ terms we get the following 4-dim Lagrangian terms

$$\mathcal{L}_{4,\text{kin}} = -\frac{1}{2\lambda_4^2} \sqrt{-g} \sum_{m=1}^{3} \left\{ \frac{1}{a_{m1} a_{m2}} (D_\mu \rho_a^m D^\mu \rho^a m + D_\mu b_a^m D^\mu b^a m) \right\}$$

$$- \frac{1}{3} \sum_{A=0}^{6} \frac{1}{a_{m1} a_{m2} a^A} \partial_\mu a^A (\rho_a^m D^\mu \rho^a m + b_a^m D^\mu b^a m)$$

$$- \frac{1}{(a_{m1})^2 a_{m2}} \rho_a^m \left( \partial_\mu \nu^{m1} D^\mu b^a m + \partial_\mu a^{m1} D^\mu \rho^a m \right)$$

$$- \frac{1}{a_{m1} (a_{m2})^2} \rho_a^m \left( \partial_\mu \nu^{m2} D^\mu b^a m + \partial_\mu a^{m2} D^\mu \rho^a m \right) \right\},$$

$$\mathcal{L}_{4,\text{gauge}} = -\frac{1}{4\lambda_4^2} \sqrt{-g} \left( a^0 F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \right),$$

$$\nu = \frac{1}{4\lambda_4^2 a^0} \sqrt{-g} f^{a}_{bc} f_{a b c e d e} \sum_{m,n,p=1}^{3} \epsilon_{mnp} \frac{1}{a_{m1} a_{n2} a_{p1} a_{p2}} \left( \rho_{b_m} \rho_{d_n} \rho_c \rho_{e_p} + \rho_{b_m} \rho_{d_n} \rho_{e_p} + b_{b_m} b_{d_n} \rho_c \rho_{e_p} + b_{b_m} b_{d_n} \rho_c \rho_{e_p} \right)$$
The full N=1 theory in the neighborhood of an isolated singularity

Kahler potential

\[ K = \frac{7}{\kappa_4^2} \ln 2 - \frac{1}{\kappa_4^2} \sum_{A=0}^{6} \ln(\tilde{T}^A + \tilde{T}^A) + \frac{1}{4\lambda_4^2} \sum_{m=1}^{3} \frac{(C_a^m + \bar{C}_a^m)(C^{am} + \bar{C}^{am})}{(\tilde{T}^m + \tilde{T}^m)(\tilde{T}^{m1} + \tilde{T}^{m1})(\tilde{T}^{m2} + \tilde{T}^{m2})} \]

Gauge kinetic function

\[ f_{ab} = \frac{1}{\lambda_4^2} \tilde{T}^{i0} \delta_{ab} , \]

Superpotential

\[ W = \frac{\kappa_4^2}{24\lambda_4^2} f_{abc} \sum_{m,n,p=1}^{3} \epsilon_{mnp} C^{am} C^{bn} C^{cp} . \]

D-terms

\[ D_a = \frac{2i\kappa_4^2}{\lambda_4^2} f_{abc} \sum_{m=1}^{3} \frac{C^{bm} \bar{C}^{cm}}{(\tilde{T}^{m1} + \tilde{T}^{m1})(\tilde{T}^{m2} + \tilde{T}^{m2})} . \]

where

\[ \tilde{T}^A = T^A - \frac{1}{24\lambda_4^2} (T^A + \bar{T}^A) \sum_{m=1}^{3} \frac{C_a^m \bar{C}^{am}}{(T^{m1} + \bar{T}^{m1})(T^{m2} + \bar{T}^{m2})} \]

\[ \lambda_4^2 = (4\pi)^{4/3} \frac{v_7^{1/3}}{v_3^{(\tau)}} \kappa_4^{2/3} \]
Relationship to N=4 super Yang-Mills theory

- This $G_2$ compactification clearly has N=1 SUSY. However, if we neglect the gravity sector (that is, hold constant the geometric moduli, $T^A$), the remaining theory is N=4 SYM, (this makes sense because we are compactifying 7-dim SYM on a 3-torus). We can re-write our results in N=4 language.

- The N=4 SYM Lagrangian

$$\mathcal{L}_{N=4} = -\frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu} + \frac{\theta}{64\pi^2} \varepsilon^{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma} - \frac{1}{2} \left( D_\mu A^a_m D_\mu A^m_a - \frac{1}{2} D_\mu B^a_m D_\mu B^m_a \right)$$

$$+ \frac{g^2}{4} \text{tr} \left( [A_m, A_n][A^m, A^n] + [B_m, B_n][B^m, B^n] + 2[A_m, B_n][A^m, B^n] \right).$$

This is exactly our 4-dim effective theory if we define…

$$A^m_a = \frac{1}{\lambda_4 \sqrt{a^1 a^2}} \rho^m_a$$

$$B^m_a = \frac{1}{\lambda_4 \sqrt{a^1 a^2}} b^m_a,$$

$$G^a_{\mu\nu} = F^a_{\mu\nu}, \quad g^2 = \frac{\lambda_4^2}{a^0}, \quad \theta = \frac{8\pi^2 \nu^0}{\lambda_4^2}$$
Interesting N=4 SYM features

• Montonen-Olive and S-duality
If we define
\[ \tau \equiv \frac{\theta}{2\pi} - \frac{4\pi i}{g^2} \]
\[ \tau = -\frac{4\pi i T^0}{\lambda_4^2} \]

Then action is invariant under the SL(2, Z) transformation
\[ \tau \rightarrow \frac{a\tau + b}{c\tau + d} \]
where \( ad-bc=1 \), with \( a, b, c, d \in \mathbb{Z} \). This includes S-duality
\[ \tau \rightarrow -\frac{1}{\tau} \]
Since the real part of \( T^0 \) is the volume of the 3-torus, here S-duality is manifested as T-duality!

• "Superconformal" Phase
Unbroken symmetry, in the neighborhood of the singularity.
\[ [Z^{am}, Z^{bn}] = 0 \quad \langle Z^{am} \rangle = 0 \]

• "Coulomb" Phase
Spontaneously broken symmetry, blowing up the singularities
\[ [Z^{am}, Z^{bn}] = 0 \quad \langle Z^{am} \rangle \neq 0 \]
Overview: What we did so far

• Considering M-theory compactifications, we are lead naturally to G2 spaces
• We need singular spaces –ADE type singularities
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• “Matching up” to the smooth $G_2$ case.
Co-dimension 7 singularities

• In order to incorporate charged chiral matter, we must intersect the co-dimension 4 and 7 singularities
• The fixed plane of the co-dim 4 singularity must intersect the tip of the cone (co-dim 7) singularity.
• No compact examples are known of G2 spaces with conical singularities
• How to get them to intersect?
The goal…

• Be able to write an explicit M-theory effective action in the neighborhood of the two intersecting singularities

\[
S = \frac{1}{\kappa^2} \int_{M^{11}} dx^{11} \sqrt{-g} (R + \ldots) + \frac{1}{\lambda^2} \int_{M^{11}} \delta(x^4) \text{(orbifold singularity)} \\
+ \frac{1}{\rho^2} \int_{M^{11}} \delta(x^7) \text{(conical singularity)}
\]

This work is in progress…
Further Directions and Applications

• Compactify on other $G_2$ spaces with Compact subspace different from $T^3$. (Local $N=1$ SUSY?)
• Generalize the procedure for other ADE singularities
• M-theory on K3 with ADE singularities
• Chiral matter – including co-dimension 7 singularities, $d=4$, $N=1$ matter fields
• Duality with type IIA and intersecting branes
The End
Comparison to the smooth limit

- We find that we can compare this form of the Kahler potential to the case of a smooth $G_2$ manifold where we have “blown-up” the A-type singularities.
- Physically, this corresponds to assigning VEVs to the real parts of the chiral multiplets along D-flat directions.
- Generically, symmetry is broken to $U(1)^{(N-1)}$
- We find unexpectedly that the results agree exactly with those previously found in the smooth limit (up to a choice of embedding the $U(1)^{(N-1)}$ into $SU(N)$).
- Potential applications close to (and at) the singularity. Useful for studying wrapped branes and their associated low energy physics.
Wilson lines and symmetry breaking

\[ U_\gamma = P \exp \left( -i \oint_\gamma X_a A^a_m dx^m \right) \]

- The first fundamental group of a 3-torus is \( \mathbb{Z}^3 \). This leads to

<table>
<thead>
<tr>
<th>Gauge Group</th>
<th>Residual Gauge Groups from Wilson lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU_2</td>
<td>U_1</td>
</tr>
<tr>
<td>SU_3</td>
<td>SU_2 \times U_1, U_1^2</td>
</tr>
<tr>
<td>SU_4</td>
<td>SU_3 \times U_1, SU_2 \times U_1^2, SU_2^2 \times U_1, U_1^3</td>
</tr>
<tr>
<td>SU_6</td>
<td>SU_5 \times U_1, SU_4 \times U_2^4, SU_2 \times SU_3 \times U_2^2, SU_2^2 \times U_1^3, SU_2 \times U_1^4, SU_3 \times U_1^3, SU_2 \times SU_4 \times U_1, SU_2^3 \times U_1^2, SU_2^2 \times U_1^2, SU_3 \times U_1^2, U_1^5</td>
</tr>
</tbody>
</table>

11-dim View
Compactification and Wilson lines

4-dim View
Turning on VEVs for certain directions of the scalar fields in the potential
Flux

• We can consider G- and F- flux and find Gukov-type formulas. The effect of G-flux is

\[ W = \frac{1}{4} \int_Y \left( \frac{1}{2} C + i\varphi \right) \wedge G_Y \]

Similarly, F-flux

\[ W = \frac{\kappa_4^2}{16\lambda_4^2} \frac{1}{\nu_3} \int_{T^3} \omega_{CS} \]

where

\[ \omega_{CS} = \left( F^a \wedge C_a - \frac{1}{3} f_{abc} C^a \wedge C^b \wedge C^c \right) \]

\[ C_a = \rho_{am} dx^m + i b_{am} dx^m \]

\[ F^a = dC^a + f^a_{\, bc} C^b \wedge C^c. \]