The $\mathrm{SL}(2) \times \mathbb{R}^+$ exceptional field theory and F-theory

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Motivations

- Exceptional Field Theory (EFT) is a by now an established framework that attempts to make manifest U-duality symmetries through a "generalised geometric" construction.
- There is a long list of works based on all sort of Exceptional groups related to U-duality in various dimensions. What about the simplest case of SL_2?
- In particular we wish to compare the EFT theory for SL_2 to expectations from F-theory (and M-theory).
- Through thinking of F-theory as an EFT can we get something new, both in terms of technical computation and conceptually novel ideas?

Outline

- Basic Features of IIB and F-theory
- Features of EFT, basic field content and local symmetries
- Construction of the $\mathrm{SL}(2) \times \mathbb{R}^+$ EFT action in 12 dimensions
- IIB section and M-theory section choices
- Connection to F-theory, 7-branes solutions of EFT
- outlook

IIB in 10-d

- Massless fields are

$$\text{metric } g_{\mu\nu} \quad \text{2-forms } \begin{matrix} B_2 \\ C_2 \end{matrix} \quad \text{4-form } C_4$$

$$\text{scalars } \tau = C_0 + ie^{-\phi}$$

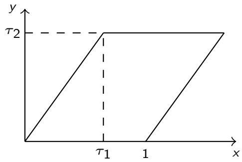
- $\mathrm{SL}(2)$ invariant

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$$

where $ad - bc = 1$.

Geometrical origin

- Shape of a torus is parametrised by a complex structure τ, identified up to $\mathrm{SL}(2,\mathbb{Z})$



$(x, y) \sim (x+1, y) \sim (x+\tau_1, y+\tau_2)$, $\tau = \tau_1 + i\tau_2$

- Introduce an auxiliary torus $T^2 \to$ 10-d IIB whose complex structure τ varies in spacetime.

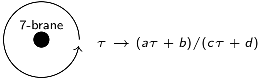
- This prompted the idea of 12-d origin of the theory? But, there is no ordinary 12-d SUGRA.

F-theory

- F-theory is a framework for analysing these fibrations. Vafa 1996

- In particular, 7-brane backgrounds. $z \in \mathbb{C}$ transverse coordinate. $\tau = \tau(z)$ describes fibration of torus (elliptic fibration).

- At D7-brane position, $\tau \sim \frac{1}{2\pi i} \log(z - z_{D_7})$. So $\tau \to i\infty$ at brane position. Paths around the brane have an $\mathrm{SL}(2)$ monodromy $\tau \to \tau + 1$.



- More general monodromies allowed $\Rightarrow$ non-perturbative (recall $\tau = C_0 + i/g_s$).
- Coincident branes $\Rightarrow$ enhanced gauge symmetries.

M-theory/IIB duality

- M-theory on $T^2(V,\tau)$ gives IIB on $\tilde{S}^1_B(\tilde{R}_B)$ with
  $\tilde{R}_B = V^{-3/4}$
  $\tau = C_0 + iE^{-\phi}$

- M-theory on T^2 in limit of vanishing torus area gives IIB in 10-dimensions. Schwarz
- A generalisation of T-duality where we exchange membrane winding with a momentum.
- We will want to keep track of membrane winding modes.

U-duality

- Reduce maximal SUGRA on a D-torus.
- Enhanced symmetry G. Scalars $\mathcal{M} \in G/H$ coset

10-D	D	G	H
9	1	$\mathrm{SL}(2) \times \mathbb{R}^+$	$\mathrm{SO}(2)$
8	2	$\mathrm{SL}(3) \times \mathrm{SL}(2)$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$
7	3	$\mathrm{SL}(5)$	$\mathrm{SO}(5)$
6	4	$\mathrm{SO}(5,5)$	$\mathrm{SO}(5) \times \mathrm{SO}(5)$
5	5	E_6	$\mathrm{USp}(8)$
4	6	E_7	$\mathrm{SU}(8)$
3	7	E_8	$\mathrm{SO}(16)$

- Can we associate scalars $\mathcal{M} \in G/H$ to geometric object? Kumar, Vafa 1996 .
- A limited number of possible geometrisations. In general, no obvious ordinary geometric interpretation.
- The answer was inspired by **generalised geometry**. The symmetry G is made manifest with an **extended space** obeying a novel sort of geometry: double field theory (T-duality) and exceptional field theory (U-duality).

Full EFT construction Hohm, Samtleben 2013 earlier work Hillmann; DSB, Perry, and many more Builds on DFT Siegel 1993; Hull, Hohm, Zwiebach and many more generalised geometry Hitchin; Gualtieri 2003; Coimbra, Strickland-Constable, Waldram and many more

Exceptional field theory: general features

- All fields, and coordinates $\in$ reps of $G \Rightarrow$ manifest duality symmetry.
- Extra coordinates associated to brane wrappings are required.
- Treats metric + forms together $\Rightarrow$ novel generalised diffeomorphism symmetry.
- Consistency $\Rightarrow$ section condition, reduces coordinate dependence.
- After imposing this condition, $\rightarrow$ 11-dimensional SUGRA or 10-dimensional IIB SUGRA.

Exceptional field theory: constructive explanation

- Take 11-d SUGRA. Split coordinates

$$x^{\hat{\mu}} \to (x^{\mu}, y^{i}) \quad \hat{\mu} = 0, \ldots, 10 \quad \mu = 0, \ldots, 10 - D \quad i = 1, \ldots, D$$

- Introduce new dual coordinates associated with brane wrappings $\tilde{y}_{ij}, \tilde{z}_{ijklm}, \ldots$.

- Then combine with the original coordinates so the extended coordinates

$$Y^{M} = (y^{i}, \tilde{y}_{ij}, \tilde{z}_{ijklm}, \ldots)$$

  form a representation of G.

Exceptional field theory: constructive explanation

- Classify dofs under splitting $\mathrm{SO}(1, 11 - D) \times \mathrm{SO}(D)$, (as if KK reduction) e.g. $g_{\mu\nu} \to g_{\mu\nu}, A_{\mu i}, g_{ij}$.
- Repackage into G representations:
  - Metric $g_{\mu\nu}$
  - Vector field ${A_\mu}^M$
  - Forms ${B_{\mu\nu}}^{(MN)}, {C_{\mu\nu\rho}}^{M\cdots}, \ldots$ (after dualisation)
  - Generalised metric for the extended space $\mathcal{M}_{MN} \in G/H$

Lets look at an example of the generalised metric on the extended space that includes the membrane winding mode coordinates $y_{\mu\nu}$ along with usual x^{μ} coordinates. No longer a simple doubling. Now the generalised tangent space is: $\Lambda^1(M) \oplus \Lambda^{*2}(M)$.
A simple example is for $E_4 = SL_5$, the generalised metric is given by:

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2} C_a{}^{ef} C_{bef} & \frac{1}{\sqrt{2}} C_a{}^{kl} \\ \frac{1}{\sqrt{2}} C^{mn}{}_b & g^{mn,kl} \end{pmatrix},$$

where $g^{mn,kl} = \frac{1}{2}(g^{mk} g^{nl} - g^{ml} g^{nk})$.

Exceptional field theory: constructive explanation

- Repackage symmetries: diffeomorphisms plus gauge

$$\delta_\Lambda V^i = \Lambda^j \partial_j V^i - V^j \partial_j \Lambda^i$$

$$\delta_\lambda C_{i_1 \dots i_p} = p \partial_{[i_1} \lambda_{i_2 \dots i_p]}$$

- Gives generalised diffeomorphisms

$$\delta_\Lambda V^M = \Lambda^N \partial_N V^M - V^N \partial_N \Lambda^M + Y^{MN}{}_{PQ} \partial_N \Lambda^P V^Q$$

Y-tensor invariant under G DSB, Cederwall, Kleinschmidt, Thompson

- These transformations are realated to the tensor hierarchy that appears in gauged supergravity.

- Also have external diffeomorphisms.

Exceptional field theory: consistency of symmetries

- Normal diffeomorphisms form an algebra under the Lie bracket.
- Generalised diffeomorphisms only close up to the constraint:

$$Y^{MN}{}_{PQ}\partial_M\mathcal{O}_1\partial_N\mathcal{O}_2 = 0 \quad Y^{MN}{}_{PQ}\partial_M\partial_N\mathcal{O} = 0$$

  known as **section condition**.
- Two inequivalent solutions not related by G: restricting coordinate dependence to
  - at most D of the Y^M coordinates, total coordinates $11 - D + D = 11$ **M-theory section**
  - at most $D-1$ of the Y^M coordinates, total coordinates $11 - D + D - 1 = 10$ **IIB section**

Exceptional field theory: action

- The (bosonic) action is entirely fixed by the (bosonic) symmetries:

$$S = \int dx dY \sqrt{|g|} \left(R(g) + \mathcal{L}_{kin} + \frac{1}{\sqrt{|g|}} \mathcal{L}_{top} + V(M, g) \right)$$

- Ricci scalar $R(g) \sim (D_\mu g)^2$
- Kinetic and gauge field terms $\mathcal{L}_{kin} \sim (D_\mu \mathcal{M})^2 + \mathcal{F}^2$
- Chern-Simons $\mathcal{L}_{top}$
- Scalar potential $V(M, g) \sim (\partial_M \mathcal{M})^2 + (\partial_M g)^2$.

- Invariant under local G by construction.

- Input section condition choice: find equivalent to 11-dimensional SUGRA/10-dimensional IIB SUGRA, under reduction.

Exceptional field theory: backgrounds

- Recall D7 brane, $\mathrm{SL}(2)$ monodromy $\tau \to \tau + 1$
- Non-geometric branes with general U-duality monodromy, e.g. 5^2_2 de Boer, Shigemori 2012

$$ds^2\Big|_{\text{transverse}} = dr^2 + r^2 d\theta^2 + \frac{H(r)}{H(r)^2 + h^2\theta^2}(dx^2 + dy^2)$$
$$B_2 = -\frac{h\theta}{H(r)^2 + h^2\theta^2} dx \wedge dy$$

- $(g, B_2)(\theta = 2\pi) =$ (T-)duality transformation of $(g, B_2)(\theta = 0)$.
- Natural description in DFT/EFT.

Cases

- The full EFT has been constructed for
  - E_8, E_7, E_6 Cederwall, Rosabal,Hohm, Samtleben 2013
  - $\mathrm{SO}(5,5)$ Abzalov, Bakhmatov, Musaev 2015
  - $\mathrm{SL}(5)$ Musaev 2015
  - $\mathrm{SL}(3) \times \mathrm{SL}(2)$ Hohm, Wang 2015
  - $+$ SUSY E_7, E_6 Godazgar, Godazgar, Hohm, Nicolai, Samtleben 2014; Musaev, Samtleben 2014

- So what about $\mathrm{SL}(2) \times \mathbb{R}^+$? Duality group for 11-dim SUGRA on T^2 / 10-dim SUGRA on S^1 / F-theory.

- Our goal: to clarify EFT vs F-theory relationship

$\mathrm{SL}(2) \times \mathbb{R}^+$ EFT

- Coordinates x^μ, $\mu = 0, \ldots, 8$ and $Y^M = (y^\alpha, y^s)$ in $\mathbf{2}_2 \oplus \mathbf{1}_{-1}$ of $\mathrm{SL}(2) \times \mathbb{R}^+$.

- Y-tensor Wang 2015
$$Y^{\alpha s}{}_{\beta s} = \delta^\alpha_\beta$$
so section condition is
$$\partial_\alpha \otimes \partial_s = 0$$

- M-theory section $\partial_s = 0$, coordinate dependence on (x^μ, y^α)
- IIB section $\partial_\alpha = 0$, coordinate dependence on (x^μ, y^s).

- Generalised diffeomorphisms
$$\delta_\Lambda V^\alpha = \Lambda^M \partial_M V^\alpha - V^\gamma \partial_\gamma \Lambda^\alpha + \partial_s \Lambda^s V^\alpha$$
$$\delta_\Lambda V^s = \Lambda^M \partial_M V^s + V^s \partial_\gamma \Lambda^\gamma - \partial_s \Lambda^s V^s$$

$\mathrm{SL}(2) \times \mathbb{R}^+$ EFT field content

- External metric $g_{\mu\nu}$
- Coset valued generalised metric $\mathcal{M}_{MN} \in \mathrm{SL}(2) \times \mathbb{R}^+/\mathrm{SO}(2)$

$$\mathcal{H}_{\alpha\beta} \in \mathrm{SL}(2)/\mathrm{SO}(2) \Rightarrow \mathcal{H}_{\alpha\beta} = \frac{1}{\mathrm{Im}\,\tau}\begin{pmatrix} |\tau|^2 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & 1 \end{pmatrix}$$

$$\mathcal{M}_{ss} \in \mathbb{R}^+$$

$\mathrm{SL}(2) \times \mathbb{R}^+$ EFT field content

- Tensor hierarchy of gauge potentials Cederwall, Edlund, Karlsson 2013; Wang 2015; DSB, Blair, Malek, Rudolph

Representation	Gauge potential	Field strength
$\mathbf{2}_1 \oplus \mathbf{1}_{-1}$	${A_\mu}^M$	${\mathcal{F}_{\mu\nu}}^M$
$\mathbf{2}_0$	${B_{\mu\nu}}^{\alpha s}$	${\mathcal{H}_{\mu\nu\rho}}^{\alpha s}$
$\mathbf{1}_1$	${C_{\mu\nu\rho}}^{[\alpha\beta]s}$	${\mathcal{J}_{\mu\nu\rho\sigma}}^{[\alpha\beta]s}$
$\mathbf{1}_0$	${D_{\mu\nu\rho\sigma}}^{[\alpha\beta]ss}$	${\mathcal{K}_{\mu\nu\rho\sigma\lambda}}^{[\alpha\beta]ss}$
$\mathbf{2}_1$	${E_{\mu\nu\rho\sigma\kappa}}^{\gamma[\alpha\beta]ss}$	${\mathcal{L}_{\mu\nu\rho\sigma\kappa\lambda}}^{\gamma[\alpha\beta]ss}$
$\mathbf{2}_0 \oplus \mathbf{1}_2$	${F_{\mu\nu\rho\sigma\kappa\lambda}}^M$	

- Interrelated gauge transformations, field strengths and Bianchi identities.

$\mathrm{SL}(2) \times \mathbb{R}^+$ EFT action

- The (bosonic) action is entirely fixed by the (bosonic) symmetries:

$$S = \int d^9 x d^3 Y \sqrt{|g|} \left(R(g) + \mathcal{L}_{kin} + V \right) + S_{top}$$

- One finds

$$\begin{aligned}
\mathcal{L}_{kin} = & -\frac{7}{32} g^{\mu\nu} D_\mu \ln \mathcal{M}_{ss} D_\nu \ln \mathcal{M}_{ss} + \frac{1}{4} g^{\mu\nu} D_\mu \mathcal{H}_{\alpha\beta} D_\nu \mathcal{H}^{\alpha\beta} \\
& - \frac{1}{2\cdot 2!} \mathcal{M}_{MN} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}^{\mu\nu N} - \frac{1}{2\cdot 3!} \mathcal{M}_{\alpha\beta} \mathcal{M}_{ss} \mathcal{H}_{\mu\nu\rho}{}^{\alpha s} \mathcal{H}^{\mu\nu\rho\beta s} \\
& - \frac{1}{2\cdot 2!4!} \mathcal{M}_{ss} \mathcal{M}_{\alpha\gamma} \mathcal{M}_{\beta\delta} \mathcal{J}_{\mu\nu\rho\sigma}{}^{[\alpha\beta]s} \mathcal{J}^{\mu\nu\rho\sigma[\gamma\delta]s} \,.
\end{aligned}$$

$$\begin{aligned}
S_{top} = \frac{1}{5!\cdot 48} \int & d^{10} x \, d^3 Y \, \varepsilon^{\mu_1 \ldots \mu_{10}} \frac{1}{4} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} \left[\frac{1}{5} \partial_s \mathcal{K}_{\mu_1 \ldots \mu_5}{}^{\alpha\beta ss} \mathcal{K}_{\mu_6 \ldots \mu_{10}}{}^{\gamma\delta ss} \right. \\
& - \frac{5}{2} \mathcal{F}_{\mu_1 \mu_2}{}^s \mathcal{J}_{\mu_3 \ldots \mu_6}{}^{\alpha\beta s} \mathcal{J}_{\mu_7 \ldots \mu_{10}}{}^{\gamma\delta} \\
& \left. + \frac{10}{3} 2 \mathcal{H}_{\mu_1 \ldots \mu_3}{}^{\alpha s} \mathcal{H}_{\mu_4 \ldots \mu_6}{}^{\beta s} \mathcal{J}_{\mu_7 \ldots \mu_{10}}{}^{\gamma\delta} \right] \,.
\end{aligned}$$

Action continued

- And

$$V = \frac{1}{4}\mathcal{M}^{ss}\left(\partial_s\mathcal{H}^{\alpha\beta}\partial_s\mathcal{H}_{\alpha\beta} + \partial_s g^{\mu\nu}\partial_s g_{\mu\nu} + \partial_s \ln|g|\partial_s \ln|g|\right)$$
$$+ \frac{9}{32}\mathcal{M}^{ss}\partial_s \ln\mathcal{M}_{ss}\partial_s \ln\mathcal{M}_{ss} - \frac{1}{2}\mathcal{M}^{ss}\partial_s \ln\mathcal{M}_{ss}\partial_s \ln|g|$$
$$+ \mathcal{M}_{ss}^{3/4}\left[\frac{1}{4}\mathcal{H}^{\alpha\beta}\partial_\alpha\mathcal{H}^{\gamma\delta}\partial_\beta\mathcal{H}_{\gamma\delta} + \frac{1}{2}\mathcal{H}^{\alpha\beta}\partial_\alpha\mathcal{H}^{\gamma\delta}\partial_\gamma\mathcal{H}_{\delta\beta}\right.$$
$$+ \partial_\alpha\mathcal{H}^{\alpha\beta}\partial_\beta \ln\left(|g|^{1/2}\mathcal{M}_{ss}^{3/4}\right)$$
$$+ \frac{1}{4}\mathcal{H}^{\alpha\beta}\left(\partial_\alpha g^{\mu\nu}\partial_\beta g_{\mu\nu} + \partial_\alpha \ln|g|\partial_\beta \ln|g|\right.$$
$$\left.\left. + \frac{1}{4}\partial_\alpha \ln\mathcal{M}_{ss}\partial_\beta \ln\mathcal{M}_{ss} + \frac{1}{2}\partial_\alpha \ln g\partial_\beta \ln\mathcal{M}_{ss}\right)\right]$$

$\mathrm{SL}(2) \times \mathbb{R}^+$ EFT reduction to SUGRA

- M-theory section $\partial_s = 0$, coordinates (x^μ, y^α).
- IIB section $\partial_\alpha = 0$, coordinates (x^μ, y^s).
- Relate EFT fields to SUGRA fields as follows (schematically):

EFT field	M-theory	IIB
$\mathcal{H}_{\alpha\beta}$	$g^{-1/2} g_{\alpha\beta}$	$\mathcal{H}_{\alpha\beta}$
$\mathcal{M}_{ss}$	$g^{-6/7}$	$(g_{ss})^{8/7}$
$A_\mu{}^\alpha$	$A_\mu{}^\alpha$	$B_{\mu s}, C_{\mu s}$
$A_\mu{}^s$	$C_{\mu\alpha\beta}$	$A_\nu{}^s$
$B_{\mu\nu}{}^{\alpha s}$	$C_{\mu\nu\alpha}$	$B_{\mu\nu},\ C_{\mu\nu}$
$C_{\mu\nu\rho}{}^{\alpha\beta s}$	$C_{\mu\nu\rho}$	$C_{\mu\nu\rho s}$
$D_{\mu\nu\rho\sigma}{}^{\alpha\beta ss}$	dual	$C_{\mu\nu\rho\sigma}$

- Theory is then equivalent to 11-dim SUGRA/IIB SUGRA.

Relationship to F-theory

- Both $\mathrm{SL}(2) \times \mathbb{R}^+$ EFT and F-theory: a 12-dimensional perspective on IIB
- EFT: Extended space has local $\mathrm{SL}(2) \times \mathbb{R}^+$ symmetry via generalised diffeomorphisms - not conventional geometry. The F-theory auxilliary torus is now part of the EFT extended geometry .
- EFT may also be reduced to M-section and when there are two isometries then we have M-theory/IIB duality.

Relationship to F-theory: M-theory/F-theory duality

- Explicit realisation of M-theory/F-theory duality. Direct mapping between fields via EFT definitions.

- Consider just extended directions,

$$\text{“}ds^2\text{”} = (\mathcal{M}_{ss})^{-3/4}\mathcal{H}_{\alpha\beta}dy^\alpha dy^\beta + \mathcal{M}_{ss}(dy^s)^2$$

  Limits $\mathcal{M}_{ss} \to 0 \Rightarrow$ M-theory directions large, $\mathcal{M}_{ss} \to \infty$ IIB direction large. $\mathcal{M}_{ss} = v^{9/7}$ gives

$$ds^2_M = v\mathcal{H}_{\alpha\beta}dy^\alpha dy^\beta$$

  and

$$ds^2_{IIB} = v^{-3/2}(dy^s)^2$$

  usual relation $R_{IIB} \sim v^{-3/4}$.

Relationship to F-theory: solutions

- e.g. 7-branes

$$\begin{aligned} ds^2_{(9)} &= -dt^2 + d\vec{x}^2_{(6)} + \tau_2 |f|^2 dz d\bar{z} \\ \text{“} ds^2_{ext} \text{”} &= \frac{1}{\tau_2}\left[|\tau|^2 (dy^1)^2 + 2\tau_1 dy^1 dy^2 + (dy^2)^2 \right] + (dy^s)^2 \end{aligned} \tag{1}$$

$$\tau = j^{-1}(P(z)/Q(z))$$

where $P(z)$ and $Q(z)$ are polynomials in z. Roots of $Q(z) \to$ brane locations.

- Sections

$$ds^2_{IIB} = -dt^2 + d\vec{x}^2_{(6)} + (dy^s)^2 + \tau_2 |f|^2 dz d\bar{z}$$

and

$$ds^2_M = -dt^2 + d\vec{x}^2_{(6)} + \tau_2 |f|^2 dz d\bar{z} + \tau_2 (dy^1)^2 + \frac{1}{\tau_2}\left(dy^2 + \tau_1 dy^1 \right)^2 \, .$$

Conclusions and outlook

- Have constructed the $\mathrm{SL}(2) \times \mathbb{R}^+$ EFT and argued it is an action for F-theory.
- Supersymmetrisation and compactification - find permitted backgrounds, derive effective actions directly.
- Higher rank Exceptional groups?
- Scherk-Schwarz F-theory? Connections to massive supergravity?
- Global issues - U-folds have been already explored. Can we consider spaces with no global choice of section so that one exchanges M-theory and F-theory sections as we move around some cycle? On a T-fold we swap, momenta and winding; the EFT version of this is to now swap the M2 winding mode mode with a momentum mode.

Thanks for the invitation and happy birthday to F-theory and to Dave!