# From flower pots to F-theory and dualities.

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D-Day, F theory@20, Caltech 2016

# Semistable degenerations of Enriques' And Hyperelliptic surfaces.

David R. Morrison, Duke Math J, 1981

Def: A semistable degeneration of surfaces is a proper holomorphic map $\pi : \mathcal{X} \to \Delta$, $\Delta$ disk s.t.

(i) $X_0 = \pi^{-1}(0)$ : reduced divisor with local normal crossing

(ii) $X_t$ is smooth; $t \neq 0$

(iii) $\mathcal{X}$ is a manifold

Theorem (Kulikov; Persson, Pinkham).

Let $\tilde{\mathcal{X}} \to \Delta$ be a s.s. deg. of surfaces with $K_{X_t} \sim \mathcal{O}_{X_t}$, $t \neq 0$, all components of $X_0$ algebraic $\Longrightarrow$

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \overset{\text{bir.}}{\dashleftarrow\dashrightarrow} & \mathcal{X} \\ \downarrow & & \downarrow \\ \Delta & & \Delta \end{array} \quad , \quad K_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}$$

semistable deg.

False if $X_t$ is Enriques:

D.R. Morrison gave analogue statement & classification.

- $X_t$: $\boxed{\text{Enriques}}$ $\quad K_{X_t} \simeq 2\mathcal{O}_{X_t}$

hyperelliptic

- Glimpses of possible applications. Mayrhofer - Lüst talks.

Terminogy:

- Pot, Flower pot (and $\mathbb{F}_4$)
- corbel

- Morrison - Stevens
- Mori - Morrison$^2$.

Classifications of (certain) terminal 3fold & 4 fold singularities.

15 years later :
( work on K3's )

**Compactifications of F-theory on Calabi-Yau threefolds, I, II:**

D.R. Morrison, C. Vafa

$\rightsquigarrow$ $\boxed{\text{cigars}}$

• Local Weierstrass equations for

$$\begin{array}{c} X_n \\ \downarrow \\ B \end{array}$$

suitable on B

$$y^2 = x^3 + f(\ )x + g(\ )$$

• If $B = \mathbb{P}^1$, $X_2 : K_3$

$$y^2 = x^3 + f_8(t)x + g_{12}(t)$$

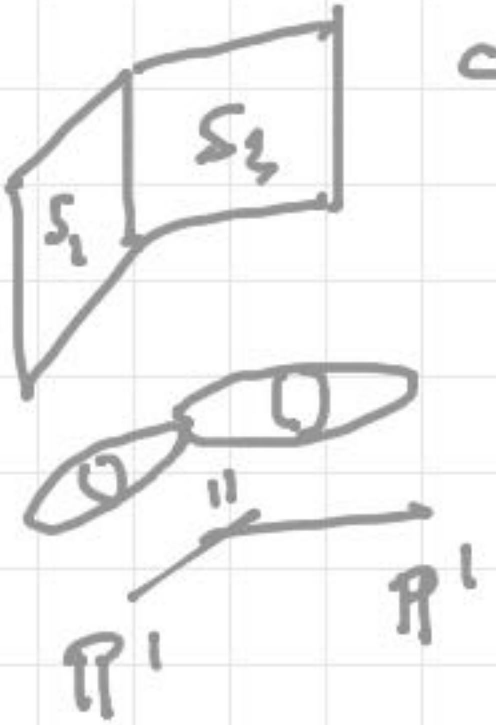
$$S_1 : y^2 = x^3 + f_4^-(t)x + g_6^-(t)$$

$$S_2 : y = x^3 + f_4^+(t) + g_6^+(t)$$

can be extended to CY3, CY4 ...

- $X = X_t$ general fiber of : $\begin{array}{c}\mathcal{X} \\ \downarrow \\ \mathbb{D}\end{array}$ ; disk

- $X_0 = S_1 \cup S_2$ s.s. degeneration.




$$\begin{array}{ccccc} \hookrightarrow & \mathcal{X} & \hookleftarrow & X_t : & K3 \\ & \downarrow & & \downarrow & \downarrow \\ & \mathbb{D} & & & = \mathbb{P}^1 \end{array}$$

two cigars

cigar

## § F-theory / Heterotic duality ($E_8 \times E_8$)

$$\begin{tikzcd} X_m \arrow[d] \arrow[dr, "K3 \text{ fibered}"] & \\ B \arrow[r] & B_{m-1} \end{tikzcd}$$

F-theory.

elliptically fibered

Heterotic

$$\begin{tikzcd} Z_{m-1} \arrow[d, "\text{elliptic}"] \\ B_{m-1} \end{tikzcd}$$

$(V_1, V_2)$ bundles w/ structure groups $(G_1, G_2)$

**Note:** on F-theory side $U(1)$s

gauge $\to$ abelian $\Leftrightarrow MW(X/B)$

gauge $\to$ $\boxed{\text{non abelian} \Leftrightarrow \left\{ \begin{array}{l} \text{codim 1} \\ \text{sing fibers} \end{array} \right.}$

Much done in case of

non-abelian gauge groups.
on F-theory.

Abelian ??

$U(1)$ ... discrete ..

Math: Mordell-Weil, $\pi_1$

Project w/ Penn group:

Cvetič - AG - Klevers - Piragua(?)-Taylor... Pretschkin - Song

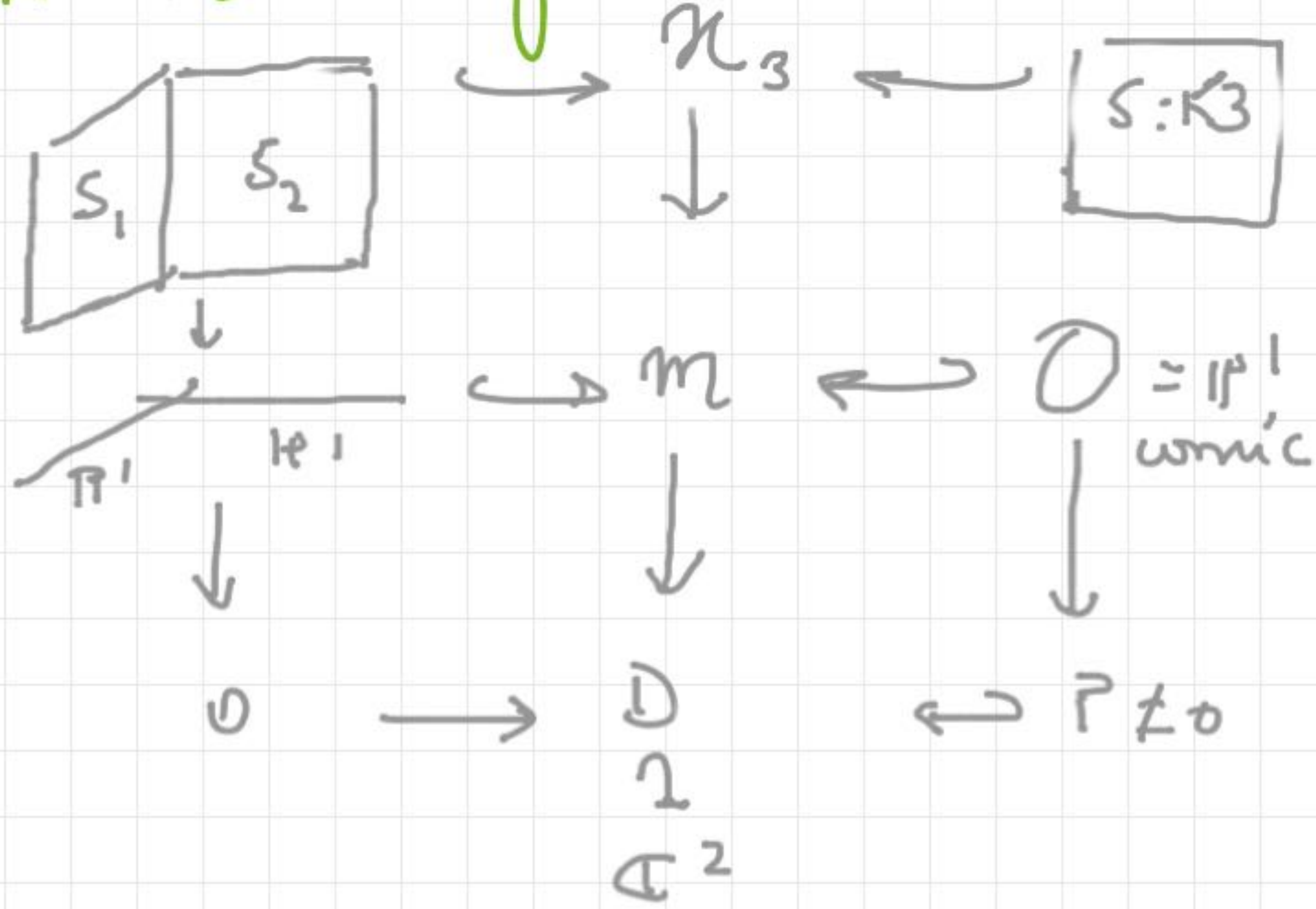
see M. Cvetič, P. Song, talks

- . -

L. Anderson's talk

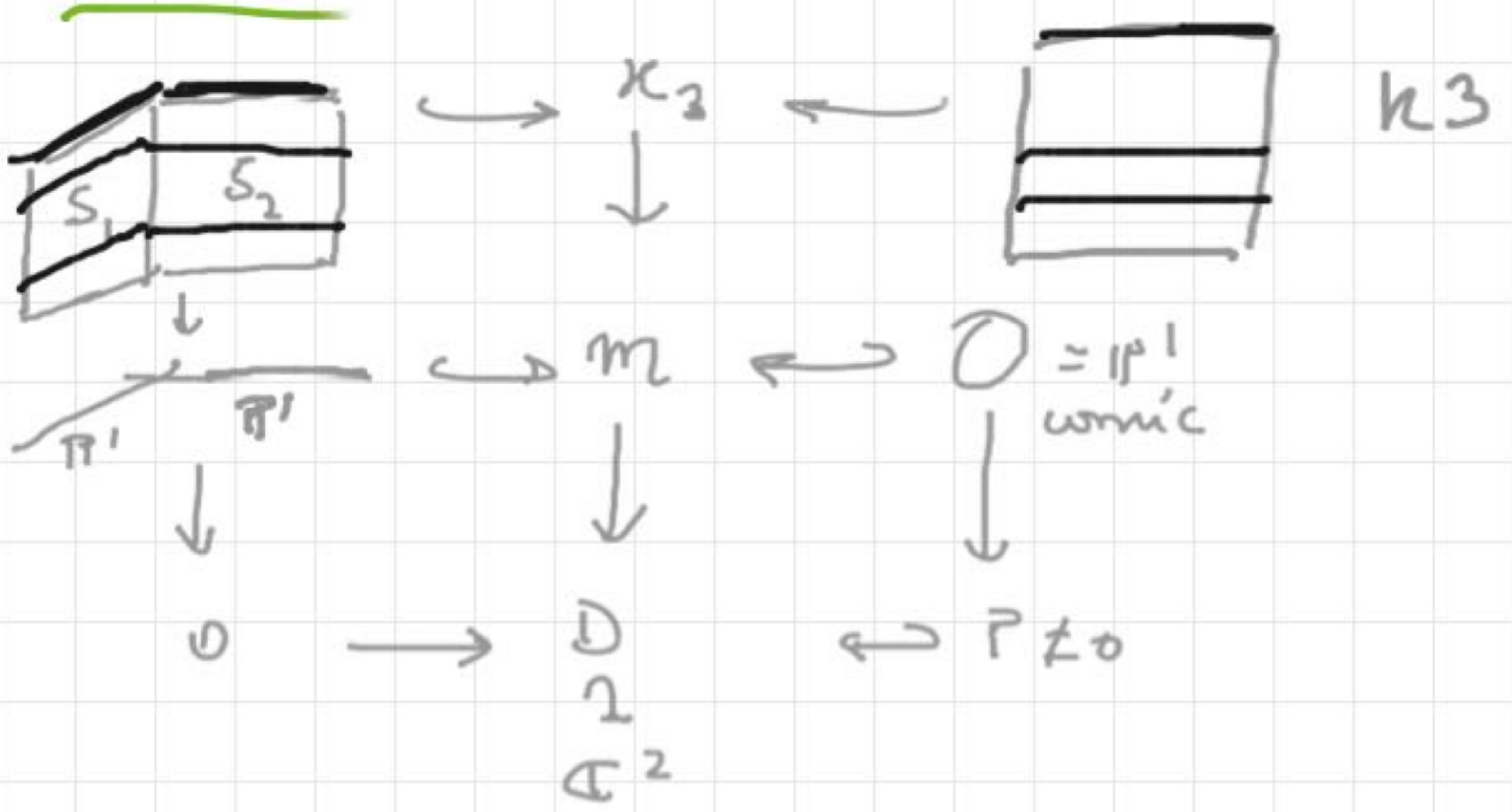
(re.: {LA, Gray, Raghuram, Taylor

Morrison - Vafa et al...




Weierstrass model: degeneration does not preserve sections

# Want:




- sections: ———
  are preserved in the degeneration

- (CGKPS) Provide construction which also works for:

- no section on F-theory

- outside $\left\{\begin{array}{l}\text{complex}\\ \text{projective}\\ \text{algebraic}\end{array}\right.$ : symplectic w/ Perm group:

- Consider CY in toric ambient spaces, with toric sections. [Perduca, 2009 thesis]

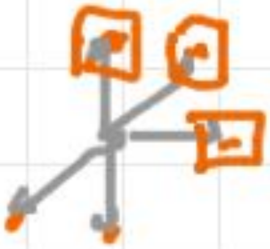
example:

(1). 

Toric fiber

□ Toric sections.

$Bl\, W\mathbb{P}(1,1,2)$ at a smooth point

(2). 

$Bl_{p_1,p_2}\mathbb{P}^3$ □ toric sections

- Toric sections are e.i. in K3 ⊂ CY)

→ (1) → rk MW = 1 : U(1)

(2) → rk MW = 2 : U(1) × U(1)

## §2. Construct toric ambient space $\mathbb{P}_\Sigma$ with fiber $\mathbb{P}_{\Sigma_2}$

<u>example:</u>

$$
\left[\begin{array}{ccc|c}
-1 & 1 & 0 & x_1 \\
-1 & -1 & 0 & \vdots \\
1 & 0 & 0 & \vdots \\
0 & 1 & 0 & \vdots \\
-1 & 0 & 0 & x_5 \\
\hline
0 & 0 & 1 & u \\
0 & 0 & -1 & v
\end{array}\right]
$$

$$
\begin{tikzcd}
\mathbb{P}_\Sigma = \mathbb{P}^1_{[u,v]} \times \mathbb{P}_{\Sigma_2} \arrow[d] & \mathbb{P}_{\Sigma_2} \arrow[l, hook'] \arrow[d] \\
\mathbb{P}^1_{[u,v]} & pt \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{P}_\Sigma \arrow[d] & X_2 : (\chi_{[-\mathbb{P}_\Sigma]} = 0) \arrow[l, hook'] \arrow[d] & E \arrow[l, hook'] \arrow[d] \\
\mathbb{P}^1 & \mathbb{P}^1 \arrow[l] & pt \arrow[l]
\end{tikzcd}
$$

- For $X$ general : $h^{11}(X_2) = 5$
- rk $MW(X_2) = 1$
- two $I_2$ fibers.

$U(1)$

$SU(2) \times SU(2)$

↑

gauge

( ~ , Bruzzo)

- Build a toric s.o. degeneration of $\mathbb{P}_\Sigma$ which preserve the sections.

- Inspired by symplectic cut techniques.

• "Predictions" by Candelas-Font
Berglund-Nouyn.

For toric F-theory / Het duality
with fiber: $WP(123)$
(no UCAs).

Back to the future
( F theory 2015 ! )

- The above degeneration is different than the Weierstrass degeneration:
  ( F-theory 2015 )
- Diagram does not commute.
  [ see: Mirjam Cvetič's talk ]
-

- Grimm - Rapter.

- Extend to higher dim. CY
  ( $n=3$, $n=4$ ).

In the paper: general
analysis.

( Reflexive Newton Polytopes )

§3: Take $X$ non-general, CY.
and analyze dualities:

- Gauge group match:
  long list of examples:
  some significative ones in
  the paper

See: M. Cvetič - P. Song talks

## Features:

- "F-theory" does not need to have a section.
  (work in progress w/ Cvetič-Poretshkin)
- "F-theory" does not need to be algebraic, complex.
- '$\mathcal{X}$: total space of degeneration does not have to be smooth.

$\rightsquigarrow$ singularities

# Birthday Riddle (1)

"This paper gives the reader an excellent chance to see a natural and straightforward question taken on, solved and exposed by a competent author."

( from Math Review )

- The birational geometry of surfaces with rational double points. (1985)

R. Miranda.

Towards understanding (up to birational equivalence).

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & X' \arrow[d] \\ B \arrow[r, dashed] & B' \end{tikzcd}$$

$X$, CY 4-fold

- Techinque of DRM '85 paper:
  "Weighted blow ups"

(not in this language)

- Natural in toric context.

  e.g.

- In higher dim:
  exceptional divisors are WP

- Weighted blow ups occur:
  $B \longrightarrow B'$ ; $\dim B, B' = 3, 4$

# Birthday riddle ②

**Theorem (AG):**

$$\begin{tikzcd} X \arrow[d] \\ B \end{tikzcd}$$

elliptic, CY, no multiple fibers

$\dim X = 3$

$\Longrightarrow$

$$\exists \quad \begin{tikzcd} \bar{X} \arrow[d] & X \arrow[l, dashed] \arrow[d] \\ \bar{B} & B \arrow[l, dashed] \end{tikzcd}$$

Birational equivalent

$\bar{B}$, smooth

**Proof:**

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[r, dashed] \arrow[d] & \bar{X} \arrow[d] \\ X \arrow[d] & \tilde{B} \arrow[dl] \arrow[r] & \bar{B} \\ B & & \end{tikzcd}$$

- Discriminant on $\hat{B}$ has s.n.c.
- No exceptional graph : $\langle n, q \rangle$ H.S.

- Question:

  Why these Hirzebruch's strings do not appear? Physics??

- Insights from SCFT?

- Hints for CY 4-folds?

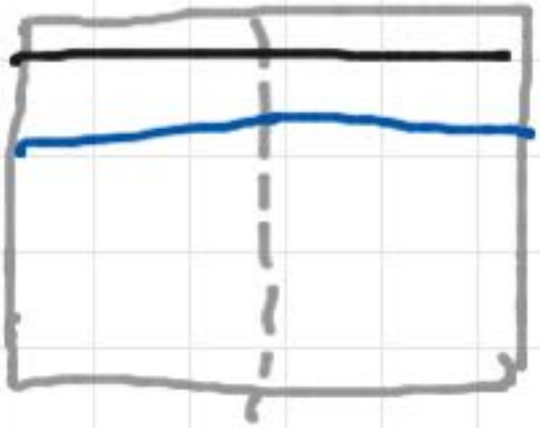
ngredients in the analysis:
Dictionary: Math/Physics.

F-theory gauge groups

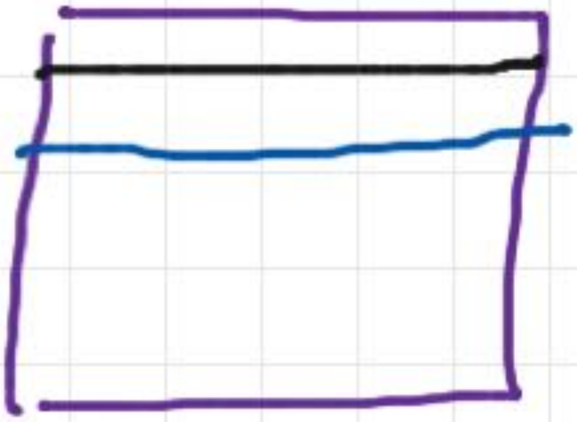
structure group of heterotic bundles.

spectral cover.

centralizers in $E_8 \times E_8$



$E_7$ , $E_7$



$E_7 \times E_7 \times U(1)$

$(\overline{\mathrm{III}}, \overline{\mathrm{IV}})$

.. structure group $U(1) \times U(1)$.

⇝ • Heterotic gauge group.

$$E_7 \times \underbrace{U(1) \times E_7 \times U(1)}_{\downarrow} \quad \text{massive}$$

$U(1)$ massless.

Stückelberg mechanism -

i.e. $E_7 \times E_7 \times U(1)$

match w/ F-theory.