Strong coupling and geometrically non-Higgsable seven-branes

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Based on:

- [J.H.] and also [Grassi, J.H., Shaneson], to appear.
- 1506 with Taylor and 1409 with Grassi, Shaneson, Taylor.
- 1306, 1402, 1410 with Grassi, Shaneson.

Motivation

A major part of the F-theory landscape.

Non-Higgsable seven-branes (NH7) are extremely common.
General arguments and large datasets as evidence.

- 6d: $\mathbb{F}_n$, NH7 on $-n$-curve for $n \geq 3$. [Morrison, Vafa]
- 6d: 65,000 toric B_2. All but 16 have NH7. 16 are weak Fano. [Morrison, Taylor]
- 4d: $\mathbb{P}^1$-bundles over toric surf., 98.3% of 100,000 B w/ NH7. [J.H., Taylor].
- 4d: Toric Monte Carlo. All have NH7 after "thermalization." [Taylor, Wang].

They're interesting.

Gauge theoretic aspects:

- 6d: Possibilities classified and understood. [Morrison, Taylor] NH b/c either pure SYM or $\mathbf{1/2}$-hyper in pseudoreal-rep. e.g. $\mathbf{56}$ of $\boldsymbol{E_6}$, $(\mathbf{7}, \mathbf{2})$ of $\boldsymbol{G_2 \times SU(2)}$.
- 4d: Much less understood from GT perspective.

Strongly coupled aspects:

- All but one type of NH7 strongly coupled in $\boldsymbol{g_s}$.
- Corollary: will study general $\boldsymbol{g_s \sim O(1)}$ region away from brane.

Outline

Goal: learn as much as we can about NH7.
Here: study fibers that may give NH7 and associated physics.

- Geometrically non-Higgsable Seven-branes
- Axiodilaton Profiles and Strong Coupling
- Non-Perturbative SU(3) and SU(2) Theories
- D3-branes and Dyons as Junctions

Geometrically NH7

- » $y^2 = x^3 + fx + g$. NH7 Along $D \subset B$ if f, g forced to vanish.
- » Fibers may only be $II, III, IV, I_0^*, IV^*, III^*, II^*$. We will study these.
- » Single factors: $E_8, E_7, E_6, F_4, SO(8), SO(7), G_2, SU(3), SU(2)$. Intersect. factors limited.
- » No $SO(10)$, $SU(5)$ or $SU(N > 3)$.

Type	$ord_z(f)$	$ord_z(g)$	$ord_z(\Delta)$	singularity	nonabelian symmetry algebra	monodromy	order
I_0	≥ 0	≥ 0	0	none	none	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
I_n	0	0	$n \geq 2$	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor n/2 \rfloor)$	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	∞
II	≥ 1	1	2	none	none	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6
III	1	≥ 2	3	A_1	$\mathfrak{su}(2)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4
IV	≥ 2	2	4	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3
I_0^*	≥ 2	≥ 3	6	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2
I_n^*	2	3	$n \geq 7$	D_{n-2}	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	∞
IV^*	≥ 3	4	8	E_6	$\mathfrak{e}_6$ or $\mathfrak{f}_4$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3
III^*	3	≥ 5	9	E_7	$\mathfrak{e}_7$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4
II^*	≥ 4	5	10	E_8	$\mathfrak{e}_8$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6
non-min	≥ 4	≥ 6	≥ 12	does not appear for supersymmetric vacua			

Axiodilaton Profiles

The Axiodilaton and the J-invariant

- τ is important, encoding g_s and seven-brane monodromy.
- $J(\tau)$ is an $SL(2,\mathbb{Z})$ invariant function, holomorphic away from $\tau = i\infty$, where it has a simple pole.
- $\tau \sim log(z)/2\pi i$ near D7-brane, I_1 fiber. More generally? Fact: $J(i) = 1$, $J(e^{2\pi i/3}) = J(e^{\pi i/3}) = 0$, gives $g_s = 1$, $g_s = \sqrt{2}/3$.
- In a Weierstrass model it is easy to compute J. $J = \frac{4f^3}{\Delta}$ with $\Delta = 4f^3 + 27g^2$. Central to Sen's limit.
- Can we invert J to get τ?

Can we invert J to get τ?

$$\tau = i\,\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}, \qquad J = \frac{4\,(1-x(1-x))^3}{27\,x^2(1-x)^2},$$

$$q = exp\left(-\pi\frac{{}_2F_1(1/2,1/2,1,1-x)}{{}_2F_1(1/2,1/2,1,x)}\right)$$

- Jacobi's method: solve for x, get six values of τ.
- Recall the q-expansion $J = \dfrac{1}{q} + 744 + 196884q + \dots$
- We want to see seven-brane monodromy on τ.
  It's always order < 6. Jacobi's method seems like overkill.
  Ramanujan has other $q - x$ that give simpler inversions.

I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen.

'No', he replied, 'it is a very interesting number.' It is the smallest number expressible as the sum of two cubes in two different ways.'

– G.H. Hardy

The Birth of the Taxicab Numbers

The n^{th} *taxicab number* is the smallest number that can be expressed as a sum of two positive cubes in n distinct ways.

The first 6 are known, and the first three are:

$$
\begin{aligned}
\mathrm{Ta}(1) &= 2 = 1^3 + 1^3 \\
\mathrm{Ta}(2) &= 1729 = 1^3 + 12^3 \\
& = 9^3 + 10^3 \\
\mathrm{Ta}(3) &= 87539319 = 167^3 + 436^3 \\
& = 228^3 + 423^3 \\
& = 255^3 + 414^3
\end{aligned}
$$

I know of no applications of these in F-theory.

Ramanujan wrote many of his results in a series of four notebooks, often without proofs.

Number theorists have been proving them for many years, even into the present day.

Investigation into the results we'll use began in 1988 with Venkatachaliengar, with subsequent work by Cooper, Berndt, Chen, Schultz, and others.

They are useful in F-theory

A hammer in search of nails.

Ramanujan's Alternative Bases (1914) and F-theory

$$q_r := exp\left(\frac{-2\pi}{\sqrt{r}} \frac{{}_2F_1(a_r, 1-a_r, 1, 1-x_r)}{{}_2F_1(a_r, 1-a_r, 1, x_r)}\right)$$

$$J = \frac{1}{4\,x_1(1-x_1)} = \frac{(1+3x_2)^3}{27\,x_2(1-x_2)^2} = \frac{(1+8x_3)^3}{64\,x_3(1-x_3)^3}.$$

- Ramanujan showed different q and x are useful.
  $a_1 = 1/6, a_2 = 1/4, a_3 = 1/6$. Using Cooper's notation.
- F-theory: pick $p \in B$, solve J, compute $\tau = \frac{\log q}{2\pi i}$.
- Note: now solve quadratic, cubic, quartic, not sextic.
- One computation: go near 7-brane, see τ and monodromy.

Framing the problem

- Near fibers that can be NH, write $f = z^n F$, $g = z^m G$, $A = 27G^2/4F^3$.
- Degeneracy in local J structure. [Kodaira]
- Monodromy order: 2 for III, III^* 3 for II, II^*, IV, IV^*.
- Expect, e.g., two τ values should suffice for III, III^*.

Fiber	J	$J\|_{z=0}$
II	$\frac{z}{A+z}$	0
III	$\frac{1}{1+Az}$	1
IV	$\frac{z^2}{A+z^2}$	0
I_0^*	$\frac{1}{1+A}$	$\frac{1}{1+A}$
IV^*	$\frac{z}{A+z}$	0
III^*	$\frac{1}{1+Az}$	1
II^*	$\frac{z^2}{A+z^2}$	0

Example: III and III^* Fibers

$$J(III) = J(III^*) = \frac{1}{1+Az}$$

$$x_1 = \frac{1 \pm \sqrt{1-1/J}}{2}$$

- Solve quadratic $4x_1(1-x_1)J - 1 = 0$ for this J.
- Explicit $\tau_\pm(z)$ near type III, III^* seven-branes, where $g_s = 1$, Loop around brane maps $\alpha_+ \to \alpha_-$, $\tau_+ \to \tau_- = -1/\tau_+$.
- Monodromy completely explicit with these techniques.

$$x_1 = \frac{1 \pm i\sqrt{Az}}{2} =: \alpha_\pm$$

$$\tau_\pm = i\,\frac{{}_2F_1(\frac{1}{6},\frac{5}{6},1,\alpha_+)}{{}_2F_1(\frac{1}{6},\frac{5}{6},1,\alpha_-)}$$

Example: II, II^*, IV, IV^* Fibers

$$J(II) = J(IV^*) = \frac{z}{A+z}, \qquad J(II^*) = J(IV) = \frac{z^2}{A+z^2}$$

- Solve $27(J-1)x_2^3 - 27(J+2)x_2^2 + 9(3J-1)x_1 - 1 = 0$ for J's.
- Recall III, III^*, $J=1$, one root goes to infinity.
- $\tau_{1,2,3}(z)$ near each seven-brane, order 3 monodromy explicit, even in expression for x_2.

$$x_2 = -\frac{1}{3} - \frac{2(2A^2)^{1/3}}{3A} e^{\frac{2\pi i n}{3}} z^{1/3} - \frac{4}{3(2A^2)^{1/3}} e^{\frac{4\pi i n}{3}} z^{2/3} - \frac{1}{A} z + O(z^{4/3}), \qquad n \in \{0,1,2\}$$

F-theory is always strongly coupled

$$J = \frac{4f^3}{4f^3 + 27g^2} \qquad x_1 = \frac{1}{2} \pm \frac{3i\sqrt{3}g}{4f^{3/2}} =: \alpha_\pm \qquad \tau_\pm = i\,\frac{{}_2F_1(\frac{1}{6},\frac{5}{6},1,\alpha_+)}{{}_2F_1(\frac{1}{6},\frac{5}{6},1,\alpha_-)}$$

- Every Weierstrass model has two special loci: $f = 0$ and $g = 0$.
- Consider $g = 0$, has $J = 1$. Use quadratic theory to solve.
- Notice: g, not $\sqrt{g}$, no monodromy in g-loop, as expected.
- No brane, but $g_s = 1 + .68542\, Im(g/f^{3/2}) + \ldots$ (exact Γ-expr).
- Similar: use cubic theory for $f = 0$ locus, strongly coupled.

But what about Sen's limit?

$$f = Cn - 3h^2, \quad g = h(Cn - 2h^2), \quad J = \frac{4\left(Cn - 3h^2\right)^3}{C^2 n^2 \left(4Cn - 9h^2\right)}, \quad x_1 = \frac{1}{2} \pm \frac{3\sqrt{3}h\left(Cn - 2h^2\right)}{4\left(3h^2 - Cn\right)^{3/2}}$$

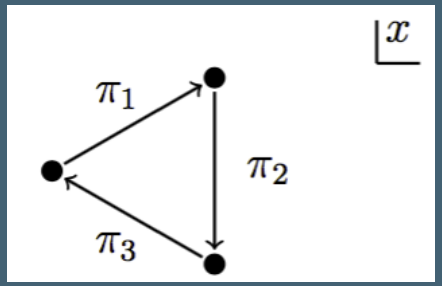
- See: g comp $h = 0$ is $g_s \sim O(1)$ region between 2 branes that collapse to O7. Only full collapse if $g_s = 0$, $O(1)$ region otherwise. Solve for τ explicitly, nice power series in h.
- Other g component? Will do for paper.
- General: $g = 0$ and $f = 0$ strongly coupled unless strict $C = 0$.

Non-perturbative SU(3) and SU(2)

String Junctions

- At varying / strong coupling IIb exhibits *string junctions*. [Schwarz] [Gaberdiel, Zwiebach] [Dewolfe, Zwiebach]
- Rigorously: these are relative homology cycles. $H_2(X, E_p)$ Intimately connected to deformations and Higgsing. [Grassi, J.H., Shaneson]
- Small deformation (locally *or* globally), see W-bosons.
- Also: see D3-7 states, codimension 2 matter, etc.

Deformations, Junctions, and Type *III*



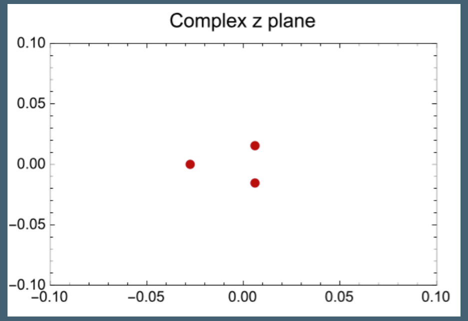
- Explicit deformation of type III gives above picture of S^2's.
- Gen. junction: boundary in $\pi^{-1}(p) = E_p$, p the central point.
- Here: boundary trivial, two-cycle in $H_2(X, \mathbb{Z})$. Give $W_\pm$.
- Holds for generic member of **family** of elliptic surfaces along deformed seven-brane. Any Kodaira fiber. All roots explicit.

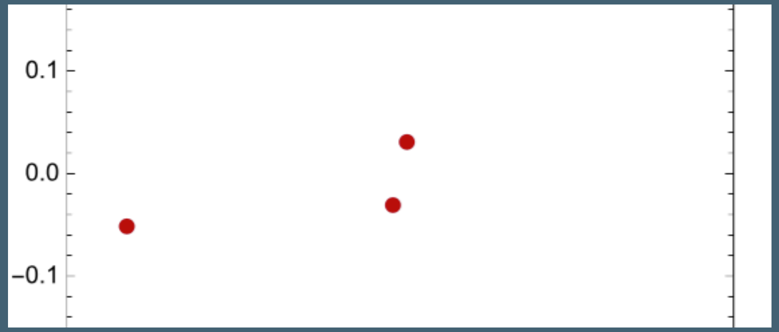
The proof is also in the pudding.

#	Simple Root	Simple Root	Simple Root	Simple Root
1	(0,0,0,−1,−1,−1)	(0,0,−1,0,0,1)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)
2	(0,0,0,1,1,1)	(0,0,−1,−1,−1,0)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)
3	(0,0,1,0,0,−1)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)
4	(0,0,1,0,0,−1)	(0,0,0,1,1,1)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)
5	(0,0,1,1,1,0)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)
6	(0,0,1,1,1,0)	(0,0,0,−1,−1,−1)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)
7	(0,1,0,0,−1,0)	(0,0,0,−1,−1,−1)	(0,−1,−1,0,1,1)	(−1,0,1,1,0,−1)
8	(0,1,0,0,−1,0)	(0,0,0,1,1,1)	(0,−1,−1,−1,0,0)	(−1,0,1,1,0,−1)
9	(0,1,0,0,−1,0)	(0,0,1,0,0,−1)	(0,−1,−1,−1,0,0)	(−1,−1,0,1,1,0)
10	(0,1,0,0,−1,0)	(0,0,1,0,0,−1)	(0,0,0,1,1,1)	(−1,−1,−1,0,0,0)
11	(0,1,0,0,−1,0)	(0,0,1,1,1,0)	(0,−1,−1,0,1,1)	(−1,−1,0,0,0,−1)
12	(0,1,0,0,−1,0)	(0,0,1,1,1,0)	(0,0,0,−1,−1,−1)	(−1,−1,−1,0,0,0)
13	(0,1,1,0,−1,−1)	(0,0,−1,−1,−1,0)	(0,−1,0,0,1,0)	(−1,0,0,1,0,0)
14	(0,1,1,0,−1,−1)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)	(−1,−1,0,1,1,0)
15	(0,1,1,0,−1,−1)	(0,0,0,1,1,1)	(0,−1,0,0,1,0)	(−1,−1,−1,0,0,0)
16	(0,1,1,0,−1,−1)	(0,0,0,1,1,1)	(0,0,−1,−1,−1,0)	(−1,−1,0,0,0,−1)
17	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(0,−1,−1,−1,0,0)	(−1,0,0,1,0,0)
18	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(0,0,−1,0,0,1)	(−1,−1,0,0,0,−1)
19	(0,1,1,1,0,0)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,−1,0,0,0,−1)
20	(0,1,1,1,0,0)	(0,0,−1,0,0,1)	(0,−1,0,0,1,0)	(−1,0,0,0,−1,−1)
21	(0,1,1,1,0,0)	(0,0,0,−1,−1,−1)	(0,−1,0,0,1,0)	(−1,−1,−1,0,0,0)
22	(0,1,1,1,0,0)	(0,0,0,−1,−1,−1)	(0,0,−1,0,0,1)	(−1,−1,0,1,1,0)
23	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(0,−1,−1,0,1,1)	(−1,0,0,0,−1,−1)
24	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(0,0,−1,−1,−1,0)	(−1,−1,0,1,1,0)
25	(1,0,−1,−1,0,1)	(0,0,0,−1,−1,−1)	(0,−1,0,0,1,0)	(−1,0,0,1,0,0)
26	(1,0,−1,−1,0,1)	(0,0,0,1,1,1)	(0,−1,0,0,1,0)	(−1,0,0,0,−1,−1)
27	(1,0,−1,−1,0,1)	(0,0,1,0,0,−1)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
28	(1,0,−1,−1,0,1)	(0,0,1,0,0,−1)	(0,0,0,1,1,1)	(−1,−1,−1,0,0,0)
29	(1,0,−1,−1,0,1)	(0,0,1,1,1,0)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
30	(1,0,−1,−1,0,1)	(0,0,1,1,1,0)	(0,0,0,−1,−1,−1)	(−1,−1,−1,0,0,0)
31	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(0,0,0,−1,−1,−1)	(−1,−1,0,1,1,0)
32	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(0,0,0,1,1,1)	(−1,−1,0,0,0,−1)
33	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(0,0,1,0,0,−1)	(−1,−1,−1,0,0,0)
34	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(0,0,1,0,0,−1)	(0,0,0,1,1,1)
35	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(0,0,1,1,1,0)	(−1,−1,−1,0,0,0)
36	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(0,0,1,1,1,0)	(0,0,0,−1,−1,−1)
37	(1,0,−1,−1,0,1)	(0,1,1,0,−1,−1)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
38	(1,0,−1,−1,0,1)	(0,1,1,0,−1,−1)	(0,−1,0,0,1,0)	(−1,−1,−1,0,0,0)
39	(1,0,−1,−1,0,1)	(0,1,1,0,−1,−1)	(0,0,0,1,1,1)	(−1,−1,−1,0,0,0)
40	(1,0,−1,−1,0,1)	(0,1,1,0,−1,−1)	(0,0,0,1,1,1)	(0,−1,0,0,1,0)
41	(1,0,−1,−1,0,1)	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(−1,−1,0,0,0,−1)
42	(1,0,−1,−1,0,1)	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(−1,0,0,1,0,0)
43	(1,0,−1,−1,0,1)	(0,1,1,1,0,0)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
44	(1,0,−1,−1,0,1)	(0,1,1,1,0,0)	(0,−1,0,0,1,0)	(−1,−1,−1,0,0,0)
45	(1,0,−1,−1,0,1)	(0,1,1,1,0,0)	(0,0,0,−1,−1,−1)	(−1,−1,−1,0,0,0)
46	(1,0,−1,−1,0,1)	(0,1,1,1,0,0)	(0,0,0,−1,−1,−1)	(0,−1,0,0,1,0)
47	(1,0,−1,−1,0,1)	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(−1,−1,0,1,1,0)
48	(1,0,−1,−1,0,1)	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(−1,0,0,0,−1,−1)
49	(1,0,0,−1,0,0)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,0,1,1,0,−1)
50	(1,0,0,−1,0,0)	(0,0,−1,0,0,1)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
51	(1,0,0,−1,0,0)	(0,0,0,1,1,1)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
52	(1,0,0,−1,0,0)	(0,0,0,1,1,1)	(0,0,−1,−1,−1,0)	(−1,−1,0,0,0,−1)
53	(1,0,0,−1,0,0)	(0,0,1,1,1,0)	(0,−1,−1,0,1,1)	(−1,0,0,0,−1,−1)
54	(1,0,0,−1,0,0)	(0,0,1,1,1,0)	(0,0,−1,0,0,1)	(−1,−1,0,0,0,−1)
55	(1,0,0,−1,0,0)	(0,1,0,0,−1,0)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
56	(1,0,0,−1,0,0)	(0,1,0,0,−1,0)	(0,−1,−1,0,1,1)	(−1,−1,0,0,0,−1)
57	(1,0,0,−1,0,0)	(0,1,0,0,−1,0)	(0,0,0,1,1,1)	(−1,−1,−1,0,0,0)
58	(1,0,0,−1,0,0)	(0,1,0,0,−1,0)	(0,0,0,1,1,1)	(−1,0,1,1,0,−1)
59	(1,0,0,−1,0,0)	(0,1,0,0,−1,0)	(0,0,1,1,1,0)	(−1,−1,0,0,0,−1)
60	(1,0,0,−1,0,0)	(0,1,0,0,−1,0)	(0,0,1,1,1,0)	(0,−1,−1,0,1,1)
61	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(0,0,−1,−1,−1,0)	(−1,−1,0,1,1,0)
62	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(0,0,−1,0,0,1)	(−1,−1,0,0,0,−1)
63	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(0,0,0,1,1,1)	(−1,−1,0,0,0,−1)
64	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(0,0,0,1,1,1)	(0,0,−1,−1,−1,0)
65	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(−1,−1,−1,0,0,0)
66	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(0,0,−1,0,0,1)
67	(1,0,0,−1,0,0)	(0,1,1,1,0,0)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
68	(1,0,0,−1,0,0)	(0,1,1,1,0,0)	(0,−1,−1,0,1,1)	(−1,−1,0,0,0,−1)
69	(1,0,0,−1,0,0)	(0,1,1,1,0,0)	(0,0,−1,−1,−1,0)	(−1,−1,0,0,0,−1)
70	(1,0,0,−1,0,0)	(0,1,1,1,0,0)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)
71	(1,0,0,−1,0,0)	(0,1,1,1,0,0)	(0,0,−1,0,0,1)	(−1,−1,0,1,1,0)
72	(1,0,0,−1,0,0)	(0,1,1,1,0,0)	(0,0,−1,0,0,1)	(−1,0,0,0,−1,−1)
73	(1,0,0,0,1,1)	(0,0,−1,−1,−1,0)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
74	(1,0,0,0,1,1)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)	(−1,0,1,1,0,−1)
75	(1,0,0,0,1,1)	(0,0,0,−1,−1,−1)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
76	(1,0,0,0,1,1)	(0,0,0,−1,−1,−1)	(0,0,−1,0,0,1)	(−1,−1,0,1,1,0)
77	(1,0,0,0,1,1)	(0,0,1,0,0,−1)	(0,−1,−1,−1,0,0)	(−1,0,0,1,0,0)
78	(1,0,0,0,1,1)	(0,0,1,0,0,−1)	(0,0,−1,−1,−1,0)	(−1,−1,0,1,1,0)
79	(1,0,0,0,1,1)	(0,1,0,0,−1,0)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
80	(1,0,0,0,1,1)	(0,1,0,0,−1,0)	(0,−1,−1,−1,0,0)	(−1,−1,0,1,1,0)
81	(1,0,0,0,1,1)	(0,1,0,0,−1,0)	(0,0,0,−1,−1,−1)	(−1,−1,−1,0,0,0)
82	(1,0,0,0,1,1)	(0,1,0,0,−1,0)	(0,0,0,−1,−1,−1)	(−1,0,1,1,0,−1)
83	(1,0,0,0,1,1)	(0,1,0,0,−1,0)	(0,0,1,0,0,−1)	(−1,−1,0,1,1,0)
84	(1,0,0,0,1,1)	(0,1,0,0,−1,0)	(0,0,1,0,0,−1)	(0,−1,−1,−1,0,0)
85	(1,0,0,0,1,1)	(0,1,1,0,−1,−1)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
86	(1,0,0,0,1,1)	(0,1,1,0,−1,−1)	(0,−1,−1,−1,0,0)	(−1,−1,0,1,1,0)
87	(1,0,0,0,1,1)	(0,1,1,0,−1,−1)	(0,0,−1,−1,−1,0)	(−1,−1,0,0,0,−1)
88	(1,0,0,0,1,1)	(0,1,1,0,−1,−1)	(0,0,−1,−1,−1,0)	(−1,0,0,1,0,0)
89	(1,0,0,0,1,1)	(0,1,1,0,−1,−1)	(0,0,−1,0,0,1)	(−1,−1,0,1,1,0)
90	(1,0,0,0,1,1)	(0,1,1,0,−1,−1)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)
91	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(0,0,−1,−1,−1,0)	(−1,−1,0,1,1,0)
92	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(0,0,−1,0,0,1)	(−1,−1,0,0,0,−1)
93	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(0,0,0,−1,−1,−1)	(−1,−1,0,1,1,0)
94	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(0,0,0,−1,−1,−1)	(0,0,−1,0,0,1)
95	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(−1,−1,−1,0,0,0)
96	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(0,0,−1,−1,−1,0)
97	(1,1,0,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
98	(1,1,0,−1,−1,0)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
99	(1,1,0,−1,−1,0)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)	(−1,0,1,1,0,−1)
100	(1,1,0,−1,−1,0)	(0,0,−1,0,0,1)	(0,−1,0,0,1,0)	(−1,0,0,0,−1,−1)
101	(1,1,0,−1,−1,0)	(0,0,0,1,1,1)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
102	(1,1,0,−1,−1,0)	(0,0,0,1,1,1)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)
103	(1,1,0,−1,−1,0)	(0,0,0,1,1,1)	(0,−1,0,0,1,0)	(−1,−1,−1,0,0,0)
104	(1,1,0,−1,−1,0)	(0,0,0,1,1,1)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)
105	(1,1,0,−1,−1,0)	(0,0,1,0,0,−1)	(0,−1,−1,−1,0,0)	(−1,0,0,1,0,0)
106	(1,1,0,−1,−1,0)	(0,0,1,0,0,−1)	(0,−1,−1,0,1,1)	(−1,0,0,0,−1,−1)
107	(1,1,0,−1,−1,0)	(0,0,1,0,0,−1)	(0,0,0,1,1,1)	(−1,0,0,0,−1,−1)
108	(1,1,0,−1,−1,0)	(0,0,1,0,0,−1)	(0,0,0,1,1,1)	(0,−1,−1,−1,0,0)
109	(1,1,0,−1,−1,0)	(0,0,1,1,1,0)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
110	(1,1,0,−1,−1,0)	(0,0,1,1,1,0)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)
111	(1,1,0,−1,−1,0)	(0,0,1,1,1,0)	(0,−1,−1,0,1,1)	(−1,−1,0,0,0,−1)
112	(1,1,0,−1,−1,0)	(0,0,1,1,1,0)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)
113	(1,1,0,−1,−1,0)	(0,0,1,1,1,0)	(0,0,−1,0,0,1)	(−1,0,0,0,−1,−1)
114	(1,1,0,−1,−1,0)	(0,0,1,1,1,0)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)
115	(1,1,0,−1,−1,0)	(0,1,1,1,0,0)	(0,−1,−1,0,1,1)	(−1,0,0,0,−1,−1)
116	(1,1,0,−1,−1,0)	(0,1,1,1,0,0)	(0,−1,0,0,1,0)	(−1,0,0,0,−1,−1)
117	(1,1,0,−1,−1,0)	(0,1,1,1,0,0)	(0,0,−1,0,0,1)	(−1,−1,0,0,0,−1)
118	(1,1,0,−1,−1,0)	(0,1,1,1,0,0)	(0,0,−1,0,0,1)	(0,−1,0,0,1,0)
119	(1,1,0,−1,−1,0)	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(−1,−1,−1,0,0,0)
120	(1,1,0,−1,−1,0)	(0,1,1,1,0,0)	(0,0,1,0,0,−1)	(0,−1,−1,0,1,1)
121	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,−1,−1,−1,0,0)	(−1,0,0,1,0,0)
122	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,−1,−1,−1,0,0)	(−1,0,1,1,0,−1)
123	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,0,−1,0,0,1)	(−1,−1,0,0,0,−1)
124	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,0,−1,0,0,1)	(−1,0,1,1,0,−1)
125	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,0,1,0,0,−1)	(−1,−1,−1,0,0,0)
126	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,0,1,0,0,−1)	(−1,0,0,1,0,0)
127	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(−1,−1,−1,0,0,0)
128	(1,1,0,−1,−1,0)	(1,0,0,0,1,1)	(0,1,1,1,0,0)	(−1,−1,0,0,0,−1)
129	(1,1,0,0,0,1)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
130	(1,1,0,0,0,1)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
131	(1,1,0,0,0,1)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,0,1,1,0,−1)
132	(1,1,0,0,0,1)	(0,0,−1,−1,−1,0)	(0,−1,0,0,1,0)	(−1,0,0,1,0,0)
133	(1,1,0,0,0,1)	(0,0,0,−1,−1,−1)	(−1,0,1,1,0,−1)	(−1,−1,−1,0,0,0)
134	(1,1,0,0,0,1)	(0,0,0,−1,−1,−1)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)
135	(1,1,0,0,0,1)	(0,0,0,−1,−1,−1)	(0,−1,0,0,1,0)	(−1,−1,−1,0,0,0)
136	(1,1,0,0,0,1)	(0,0,0,−1,−1,−1)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)
137	(1,1,0,0,0,1)	(0,0,1,0,0,−1)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
138	(1,1,0,0,0,1)	(0,0,1,0,0,−1)	(0,−1,−1,−1,0,0)	(−1,−1,0,1,1,0)
139	(1,1,0,0,0,1)	(0,0,1,0,0,−1)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)
140	(1,1,0,0,0,1)	(0,0,1,0,0,−1)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)
141	(1,1,0,0,0,1)	(0,0,1,0,0,−1)	(0,0,−1,−1,−1,0)	(−1,0,0,1,0,0)
142	(1,1,0,0,0,1)	(0,0,1,0,0,−1)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)
143	(1,1,0,0,0,1)	(0,0,1,1,1,0)	(0,−1,−1,−1,0,0)	(−1,0,0,1,0,0)
144	(1,1,0,0,0,1)	(0,0,1,1,1,0)	(0,−1,−1,0,1,1)	(−1,0,0,0,−1,−1)
145	(1,1,0,0,0,1)	(0,0,1,1,1,0)	(0,0,0,−1,−1,−1)	(−1,0,0,1,0,0)
146	(1,1,0,0,0,1)	(0,0,1,1,1,0)	(0,0,0,−1,−1,−1)	(0,−1,−1,0,1,1)
147	(1,1,0,0,0,1)	(0,1,1,0,−1,−1)	(0,−1,−1,−1,0,0)	(−1,0,0,1,0,0)
148	(1,1,0,0,0,1)	(0,1,1,0,−1,−1)	(0,−1,0,0,1,0)	(−1,0,0,1,0,0)
149	(1,1,0,0,0,1)	(0,1,1,0,−1,−1)	(0,0,−1,−1,−1,0)	(−1,−1,0,1,1,0)
150	(1,1,0,0,0,1)	(0,1,1,0,−1,−1)	(0,0,−1,−1,−1,0)	(0,−1,0,0,1,0)
151	(1,1,0,0,0,1)	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(−1,−1,−1,0,0,0)
152	(1,1,0,0,0,1)	(0,1,1,0,−1,−1)	(0,0,1,1,1,0)	(0,−1,−1,−1,0,0)
153	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,−1,−1,0,1,1)	(−1,0,0,0,−1,−1)
154	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,−1,−1,0,1,1)	(−1,0,1,1,0,−1)
155	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,0,−1,−1,−1,0)	(−1,−1,0,1,1,0)
156	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,0,−1,−1,−1,0)	(−1,0,1,1,0,−1)
157	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,0,1,1,1,0)	(−1,−1,−1,0,0,0)
158	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,0,1,1,1,0)	(−1,0,0,0,−1,−1)
159	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(−1,−1,−1,0,0,0)
160	(1,1,0,0,0,1)	(1,0,0,−1,0,0)	(0,1,1,0,−1,−1)	(−1,−1,0,1,1,0)
161	(1,1,1,0,0,0)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
162	(1,1,1,0,0,0)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
163	(1,1,1,0,0,0)	(0,0,−1,−1,−1,0)	(−1,0,0,1,0,0)	(−1,−1,0,0,0,−1)
164	(1,1,1,0,0,0)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,−1,0,0,0,−1)
165	(1,1,1,0,0,0)	(0,0,−1,−1,−1,0)	(0,−1,−1,0,1,1)	(−1,0,0,1,0,0)
166	(1,1,1,0,0,0)	(0,0,−1,−1,−1,0)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)
167	(1,1,1,0,0,0)	(0,0,−1,0,0,1)	(−1,0,0,0,−1,−1)	(−1,−1,0,1,1,0)
168	(1,1,1,0,0,0)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)	(−1,−1,0,1,1,0)
169	(1,1,1,0,0,0)	(0,0,−1,0,0,1)	(0,−1,−1,−1,0,0)	(−1,0,0,0,−1,−1)
170	(1,1,1,0,0,0)	(0,0,−1,0,0,1)	(0,−1,0,0,1,0)	(−1,0,1,1,0,−1)
171	(1,1,1,0,0,0)	(0,0,0,−1,−1,−1)	(0,−1,−1,0,1,1)	(−1,0,1,1,0,−1)
172	(1,1,1,0,0,0)	(0,0,0,−1,−1,−1)	(0,−1,0,0,1,0)	(−1,0,0,1,0,0)
173	(1,1,1,0,0,0)	(0,0,0,−1,−1,−1)	(0,0,−1,0,0,1)	(−1,0,1,1,0,−1)
174	(1,1,1,0,0,0)	(0,0,0,−1,−1,−1)	(0,0,−1,0,0,1)	(0,−1,0,0,1,0)
175	(1,1,1,0,0,0)	(0,0,0,1,1,1)	(0,−1,−1,−1,0,0)	(−1,0,1,1,0,−1)
176	(1,1,1,0,0,0)	(0,0,0,1,1,1)	(0,−1,0,0,1,0)	(−1,0,0,0,−1,−1)
177	(1,1,1,0,0,0)	(0,0,0,1,1,1)	(0,0,−1,−1,−1,0)	(−1,0,1,1,0,−1)
178	(1,1,1,0,0,0)	(0,0,0,1,1,1)	(0,0,−1,−1,−1,0)	(0,−1,0,0,1,0)
179	(1,1,1,0,0,0)	(0,1,0,0,−1,0)	(0,−1,−1,−1,0,0)	(−1,0,1,1,0,−1)
180	(1,1,1,0,0,0)	(0,1,0,0,−1,0)	(0,−1,−1,0,1,1)	(−1,0,1,1,0,−1)
181	(1,1,1,0,0,0)	(0,1,0,0,−1,0)	(0,0,0,−1,−1,−1)	(−1,−1,0,1,1,0)
182	(1,1,1,0,0,0)	(0,1,0,0,−1,0)	(0,0,0,−1,−1,−1)	(0,−1,−1,0,1,1)
183	(1,1,1,0,0,0)	(0,1,0,0,−1,0)	(0,0,0,1,1,1)	(−1,−1,0,0,0,−1)
184	(1,1,1,0,0,0)	(0,1,0,0,−1,0)	(0,0,0,1,1,1)	(0,−1,−1,−1,0,0)
185	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,−1,0,0,1,0)	(−1,0,0,0,−1,−1)
186	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,−1,0,0,1,0)	(−1,0,0,1,0,0)
187	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,0,0,−1,−1,−1)	(−1,−1,0,1,1,0)
188	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,0,0,−1,−1,−1)	(−1,0,0,1,0,0)
189	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,0,0,1,1,1)	(−1,−1,0,0,0,−1)
190	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,0,0,1,1,1)	(−1,0,0,0,−1,−1)
191	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(−1,−1,0,0,0,−1)
192	(1,1,1,0,0,0)	(1,0,−1,−1,0,1)	(0,1,0,0,−1,0)	(−1,−1,0,1,1,0)

» e.g. All D_4 root junctions, in **192** sets of simple roots. Monodromy reduction to G_2, also. [Grassi, J.H., Shaneson]

Deformation Induced Hanany-Witten





- Two param: take a for III to I_2 and ϵ for breaking.
- Puzzle: Take just $\epsilon \neq 0$, get Mercedes picture. $W_\pm$ junctions. Then $a \to \infty$, must become fundamental string.
- $a = 0$: seven-branes (p, q) charges are π_2, π_1, π_3.
- a large: seven-branes (p, q) charges are π_1, π_1, π_3. "AAC".

D3-branes

F-theory with D3 near 7-branes

N=2 theory on D3 locally near 7-b. [Banks, Douglas, Seiberg]

Review:

- $N_f = 3, 2, 1$ Seiberg-Witten near $I_{3,2,1}$, i.e. 3,2, 1 D7's.
- Near E_6, E_7, E_8, realize Minahan-Nemeschansky.
  See e.g. [Heckman, Vafa et al].
- D3-7 collisions gives additional massless states.
- BPS states from D3-7 string junctions.
  [Mikhailov, Nekrasov, Sethi] [Dewolfe, Zwiebach et al]

BPS Condition for Junctions

$$(J, J) \geq -2 + gcd(a(J))$$

[Dewolfe, Hauer, Iqbal, Zwiebach]

- They did $N_f = 0, 1, 3, 4$ SW theory.
- Found all junctions in (deformations of) those theories that satisfy this constraint. Matched known BPS spectra.
- II, III, IV are $N_f = 1, 2, 3$ at Argyres-Douglas points. Movie: saw dyonic brane come in for III! BPS string junctions?

Argyres-Douglas BPS States as Junctions

$a(J)$	Junctions
$(-1,-1)$	$(0,0,-1,-1)$ $(-1,0,0,-1)$ $(-1,-1,0,0)$
$(1,0)$	$(1,0,0,0)$ $(0,0,1,0)$ $(0,-1,1,1)$
$(0,1)$	$(1,1,-1,0)$ $(0,1,0,0)$ $(0,0,0,1)$
$(2,1)$	$(1,0,1,1)$
$(-1,-2)$	$(-1,-1,0,-1)$
$(-1,1)$	$(0,1,-1,0)$

- D3-brane near type IV fiber. Def. of [Grassi, J.H., Shaneson]. $Z=\{\pi_1,\pi_3,\pi_1,\pi_3\}$. Do junctions realize the dyons?
- Above: solutions. EM charge, non-trivial flavor reps. Match BPS spectra in max chamber. e.g. [Maruyoshi, Park, Yan].

Argyres-Douglas BPS States as Junctions

$a(J)$	Junctions
$(1,0)$	$(0,1)$
$(1,-1)$	$(-1,1)$
$(0,1)$	$(1,0)$

$a(J)$	Junctions
$(3,1)$	$(1,1,1)$
$(2,1)$	$(1,0,1)$,$(0,1,1)$
$(1,1)$	$(1,0,1)$
$(1,0)$	$(1,0,0)$,$(0,1,0)$

- II: Explicitly compute vanishing cycles π_1, π_3.
- III: AAC of Zwiebach-Dewolfe. Also works for our π_2, π_1, π_3.
- Above: solutions to II, III. EM charge, flavor reps. Match BPS spectra. e.g. [Maruyoshi, Park, Yan].

In conclusion:

- NH7 are ubiquitous in F-theory. Interesting fibers.
- Studied their τ's using Ramanujan's alternative bases. Explicit solutions in z. Monodromy automatic.
- Corollary: see $g_s \sim O(1)$ regions near $f = 0$ and $g = 0$. F-theory always strongly coupled unless $g_s = 0$.
- $III - I_2$ relationship via def. induced Hanany-Witten. String junctions become fundamental strings in I_2 limit.
- BPS states with dyon charges as junctions in A-D on D3.

Many thanks to the organizers for a wonderful conference, and happy birthday to F-theory!

Local τ's

$$\tau_\pm = i \pm B\sqrt{Az} - \frac{i}{2}B^2 Az + O(z^{3/2})$$

$$\tau_0 \simeq e^{\frac{\pi i}{3}} - \frac{.3355i}{A^{2/3}} z^{1/3} + O(z^{2/3})$$
$$\tau_1 \simeq e^{\frac{\pi i}{3}} + \frac{.2906 + .1678i}{A^{2/3}} z^{1/3} + O(z^{2/3})$$
$$\tau_2 \simeq e^{\frac{\pi i}{3}} - \frac{.2906 - .1678i}{A^{2/3}} z^{1/3} + O(z^{2/3}).$$

- Left: III and III*.
- Right: II and IV*.

Jacobi's inversion

$$y = \pi \frac{{}_2F_1(1/2, 1/2; 1; 1-x)}{{}_2F_1(1/2, 1/2; 1; x)}, \quad z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right), \quad \text{and} \quad q = \exp(-y).$$

$$x = \left(\frac{\sum_{j=-\infty}^{\infty} q^{(j+1/2)^2}}{\sum_{j=-\infty}^{\infty} q^{j^2}}\right)^4 \quad \text{and} \quad z = \left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)^2.$$

- Take x on the interval define y, z, q.
- Invert to get x and z in terms of q.

Inversion for Ramanujan, from Cooper

$$P = P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{j q^j}{1 - q^j},$$

$$Q = Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - q^j},$$

$$R = R(q) = 1 - 504 \sum_{j=1}^{\infty} \frac{j^5 q^j}{1 - q^j}.$$

$$x_r(1 - x_r) = \frac{Q^3(q) - R^2(q)}{4Q^3(q)} \quad \text{if } r = 1$$

$$\left(\frac{x_r}{1 - x_r}\right)^{r-1} = r^6 q^{r-1} \prod_{j=1}^{\infty} \frac{(1 - q^{jr})^{24}}{(1 - q^j)^{24}} \quad \text{if } r = 2, 3, \text{or } 4.$$