Extremal transitions of Calabi-Yau fourfolds in M-theory

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F-Theory at 20: Dave Day
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Outline

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M-Theory Background

- X CY3, $M[X]$
- 3-form gauge field C_3, $G = dC_3$
- G is quantized:

$$\frac{G}{2\pi} - \frac{c_2(X)}{2} \in H^4(M, \mathbf{Z})$$

- Tadpole cancellation

$$M = \frac{\chi(X)}{24} - \frac{1}{2}\int_X \frac{G}{2\pi} \wedge \frac{G}{2\pi} \in \mathbf{Z},$$

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Superpotentials

- G induces superpotential

$$W = \int_X \Omega \wedge \frac{G}{2\pi}$$

and twisted superpotential

$$\tilde{W} = \int_X J \wedge J \wedge \frac{G}{2\pi}$$

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Extremal Transitions in M-Theory

- From singular X_0, can blow up to $X^\sharp$ or deform to $X^\flat$

$$\begin{array}{ccc} & & X^\sharp \\ & & \downarrow \\ X^\flat & \rightsquigarrow & X_0 \end{array}$$

- In M-theory, need $G^\sharp$ and $G^\flat$
- $M^\sharp$ and $M^\flat$ M2-branes for tadpole cancellation
- Look for transitions for which the M2 branes are spectators—M2-branes kept away from $S = \mathrm{Sing}(X_0)$
- $M^\sharp = M^\flat$, or

$$\frac{\chi(X^\flat)}{24} - \frac{\chi(X^\sharp)}{24} = \frac{1}{2}\int_X \frac{G^\flat}{2\pi} \wedge \frac{G^\flat}{2\pi} - \frac{1}{2}\int_X \frac{G^\sharp}{2\pi} \wedge \frac{G^\sharp}{2\pi}$$

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Surfaces of A_{n-1} Singularities

- Extremal transition where $S \subset X_0$ is a smooth surface of transverse A_{n-1} singularities
- Local equation $xy = z^n$, $z \in K_S$
- If $c_2(X^\sharp)$ is even, then $G^\sharp = 0$ satisfies quantization and tadpole cancellation for suitable $M^\sharp$
- The **main result** in this talk: given any toric hypersurface X_0 such that $c_2(X^\sharp)$ is the restriction of an even class on the toric variety , we can always find a transition in M theory from $(X^\sharp, G^\sharp = 0)$ to an $(X^\flat, G^\flat)$ which satisfies quantization and tadpole cancellation.
- Furthermore, the geometric moduli of $(X^\flat, G^\flat)$ perfectly match the predictions of the low energy theory

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Constraints from M theory

- Constraints from M theory on transition $(X^\sharp, 0) \to (X^\flat, G^\flat)$:
- Quantization:

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$$\frac{1}{2}\int_X \frac{G^\flat}{2\pi} \wedge \frac{G^\flat}{2\pi} = \frac{(n+1)n(n-1)K_S^2}{24}$$

using

$$\chi(X^\flat) - \chi(X^\sharp) = (n+1)n(n-1)K_S^2$$

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Low Energy Theory

- Low energy 3D dynamics is given by a twisted dimensional reduction of $N = 1$ 7D $SU(n)$ SYM on S
- $SU(n)$ gauge theory with $p_g + q$ adjoint chirals

$$p_g = h^{2,0}(S) = h^0(S, K_S), \qquad q = h^{1,0}(S)$$

- Coulomb branch of dimension $(n-1)(p_g + q + 1)$, parametrized by vevs of gauge bosons in Cartan

$$(\phi_1, \ldots, \phi_n), \qquad \sum \phi_i = 0$$

  and the vevs of the $n-1$ $U(1)^{n-1}$-neutral scalars in each of the $p_g + q$ chirals
- Residual $S_n = W(A_{n-1})$ action on Coulomb branch
- Higgs branch dimension $(n^2 - 1)(p_g + q - 1)$

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Checks against gauge theory predictions

- $(n-1)(p_g+1)$ Columb branch moduli ($q=0$): blowup modes in Kähler moduli and

$$xy + \prod_{i=1}^{n}(z+\eta_i), \qquad \eta_i \in H^0(S, K_S), \ \sum \eta_i = 0$$

- $(n^2-1)(p_g-1)$ flat directions for $G^\flat$
- Have S_n action on Coulomb branch
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1. Extremal Transitions in M-Theory
2. Field Theory Analysis
3. Toric Case

Toric CY 4folds with A_{n-1} singularities

- N, M dual lattices of rank 5
- (Δ, Δ°) polar 5-dimensional reflexive polytopes
- $\Delta \subset M_{\mathbb{R}}, \Delta^\circ \subset N_{\mathbb{R}}$
- 0 unique interior point of $\Delta \cap M$ and of $\Delta^\circ \cap N$
- Fan $\Sigma^\sharp$ of toric variety $\mathbb{P}_\Delta$: cones over the faces of Δ°
- Highly singular, so we choose a maximal projective crepant subdivision of that fan
- A_{n-1} case: Δ° has a one-dimensional edge Γ containing $n-1$ interior lattice points
- $X^\sharp \subset X_{\Sigma^\sharp}$ anticanonical hypersurface — CY 4fold
- Removing the interior lattice points of Γ blows down $X_{\Sigma^\sharp}$ to X_{Σ_0}, and $X^\sharp$ to X_0

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Example

- Describe $\mathbf{P}(1,1,2,2,2,2)$ (cf. Ronen's talk) torically
- Take edges of Σ_0 as rows of

$$\begin{pmatrix} -1 & -2 & -2 & -2 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Have unique interior point $(0,-1,-1,-1,-1)$ of edge joining first two vertices
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Labeling Conventions

- Label vertices of Σ_0 as $v_1, \ldots, v_k$; v_1, v_2 endpoints of Γ
- For each v_j we have homogeneous coordinate x_j and divisor $D_j \subset X_{\Sigma_0}$ given by $x_j = 0$
- Interior points $v_0, v_{-1}, \ldots, v_{2-n}$ in order
- For the vertices $v_{2-n}, \ldots, v_k$ we similarly have toric divisors $D_j^\sharp \subset X^{\Sigma^\sharp}$

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- Embed $\iota : X_\Sigma \hookrightarrow X_{\Sigma^\flat}$ for suitable fan $\Sigma^\flat$: $(x_1, x_2, x_3, \ldots, x_k) \mapsto (x_1^n, x_2^n, x_1 x_2, x_3, \ldots, x_k) =: (y_0, \ldots, y_k)$
- The A_{n-1} is visible from $q_0(y) := y_0 y_1 - y_2^n = 0$
- Convenient choice for $\Sigma^\flat$: choose $m_\Gamma \in M$

$$\langle m_\Gamma, v_1 \rangle = n - 1, \ \langle m_\Gamma, v_2 \rangle = -1$$

- Take as edges of $\Sigma^\flat$, with appropriate higher-dimensional cones

$$w_0 = \left(\frac{v_1 - v_2}{n}, -(n-1) \right) ,$$
$$w_1 = (0, 1) ,$$
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- Have toric divisors $D_j^\flat \subset X_{\Sigma^\flat}$ associated with edges w_j
- $D_0^\flat + D_1^\flat \sim N D_2^\flat$
- $\iota^*(D_2^\flat) = D_1 + D_2$
- $\iota^*(D') = D_1 + \ldots + D_k, \ D' := D_2 + D_3 + \ldots + D_k$
- Choose section $g(y)$ of D' which pulls back by ι to an equation for X_0
- $\iota(X_0)$ is the complete intersection of $q(y)$ and $g(y)$
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Example

- Return to $\mathbf{P}(1,1,2,2,2,2)$. Take $m_\Gamma = (1,0,0,0,0)$
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- $K_S = K_{X_{\Sigma^\flat}}(D_0^\flat + D_1^\flat + D_2^\flat + D')|_S = \mathcal{O}_S(D_2)$
- Basis for sections of K_S correspond to interior lattice points of dual face Γ°
- p_g is the number of these points.
- Relabel coordinates so these correspond to $y_3, \ldots, y_{p_g+2}$
- $\tilde{q} = y_0 y_1 - y_2^n - \sum_{j=0}^{n-2} h_{n-j}(y_3, \ldots, y_{p_g+2}) y_2^j$, $\deg h_j = j$

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Flat Directions

- Will exhibit $G^\flat$ with the following smoothings of X_0 as flat directions

$$\tilde{q} = y_0 y_1 - \det\left(y_2 I_n + M(y_3, \ldots, y_{p_g+2})\right)$$

  with $M(y)$ a traceless $n \times n$ matrix of linear forms
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- Specialize to $n = 2$ for simplicity
- $$M(y) = \begin{pmatrix} \ell_{11}(y) & \ell_{12}(y) \\ \ell_{21}(y) & -\ell_{11}(y) \end{pmatrix}$$
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- $G^\flat$ is an algebraic class, hence of type $(2, 2)$
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Tadpole

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- Recall: S is the complete of divisors in the classes $D_0^\flat, D_1^\flat, D_2^\flat, D'$
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$$\int_{X^\flat} \frac{G^\flat}{2\pi} \wedge \frac{G^\flat}{2\pi} = \frac{(n+1)n(n-1)}{12} \int_{X^\flat_\Sigma} D^\flat_0 D^\flat_1 (D^\flat_2)^3 D'$$

- Recall: S is the complete of divisors in the classes $D^\flat_0, D^\flat_1, D^\flat_2, D'$
- K_S is the restriction of D_2
- Integral on the right is just K_S^2
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Quantization

- $$0 \to \mathcal{O}_{X_{\Sigma^\sharp}}^{k+n-6} \to \bigoplus_{i=2-n}^{k} \mathcal{O}_{X_{\Sigma^\sharp}}(D_i^\sharp) \to T_{X_{\Sigma^\sharp}} \to 0$$
- Leading to $c_2(X^\sharp) = \sum_{i<j} D_i^\sharp D_j^\sharp$
- Similarly $c_2(X^\flat) = \sum_{i<j} D_i^\flat D_j^\flat$
- $$\frac{G^\flat}{2\pi} = \frac{1}{2}(T_1 - T_2) \equiv \frac{1}{2}(T_1 + T_2) \pmod{\mathbf{Z}}$$
- But $T_1 + T_2$ is the complete intersection $y_2 + \ell_{11}(y) = \tilde{q}(y)$ in F, cohomology class $(D_2^\flat)^2$
- We can replace $G^\flat/(2\pi)$ with $(D_2^\flat)^2/2$
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HAPPY BIRTHDAY DAVE!