Calabi-Yau manifolds realizing symplectically rigid monodromy tuples

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(joint work with Chuck Doran,
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Irreducible Calabi-Yau operators $L \in \mathbb{C}[\theta, t]$

Irreducible **Calabi-Yau operators** $L \in \mathbb{C}[\theta, t]$ satisfy:

1. $t = 0$ point of maximal unipotent monodromy.
2. L is self-dual: $\exists\, g \in \mathbb{Q}(t)^{\text{alg}} : Lg = gL^*$ $\Rightarrow$ $\text{Aut}(L/\mathbb{C}(t)) \subset \text{Sp}_n(\mathbb{C})$.
3. P has N-integral holomorphic solution at $t = 0$.
4. Further integrality properties (q-coordinate of mirror-map, Yukawa-coupling, instanton numbers).

CY-operators of order four intend to axiomatize properties of the Picard-Fuchs operator and periods for a family $\pi : X \to \mathbb{P}^1$ *of Calabi-Yau threefolds, which has a large structure limit and* $h^{2,1} = 1$ *on its generic fibers.*

But families of this type are quite difficult to find!

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4. Further integrality properties (q-coordinate of mirror-map, Yukawa-coupling, instanton numbers).

For **rank** 4, G. Almkvist et al. have a list of 565 CY-operators satisfying properties 1), 2), and 3).

Are there corresponding one-parameter families of **Calabi-Yau threefolds** *with $h^{2,1} = 1$ that realize L as Picard-Fuchs operators?*

Irreducible Calabi-Yau operators $L \in \mathbb{C}[\theta, t]$

- The first 14 entries of the list are **hypergeometric functions** of the form (a_i certain rational numbers)

$$ {}_4F_3\left(\begin{array}{c} a_1,\ a_2,\ 1-a_2,\ 1-a_1 \\ 1,1,1 \end{array} \middle| \, t \right) . $$

- Candelas et al. ['91] computed periods (and much more) for mirror of quintic family in $\mathbb{P}^4$ ($\Rightarrow a_1 = \frac{1}{5}, a_2 = \frac{2}{5}$):

$$ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5t\, x_0 x_1 x_2 x_3 x_4 = 0 . $$

- Doran and Morgan ['06] derived all 14 classifying weight-3 VHS with deformation space $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ that resemble quintic family of Calabi-Yau threefolds; subsequently toric realizations were found by Doran et al.

What is a rigid mondromy tupel?

Fuchsian differential operator L of rank n with sing. locus $S \subset \mathbb{P}^1$
$\Leftrightarrow$ Local system $\mathbb{L}(U) := \{f \in \mathcal{O}_{\mathbb{P}^1 \backslash S}(U) | L(f) = 0\}$ of rank n,
$\Leftrightarrow$ Monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \backslash S, x_0) \to \mathrm{Gl}(\mathbb{L}_{x_0}) \cong \mathrm{Gl}_n(\mathbb{C}),$$

$\Leftrightarrow$ Monodromy tupel of rank n (up to simultaneous conjugation)

$$T = (T_1, T_2, \ldots, T_r) := \Big(\rho(\gamma_1), \rho(\gamma_2), \ldots, \rho(\gamma_r)\Big) \in \mathrm{Gl}_n(\mathbb{C})^r$$

$$\text{s.t.} \quad \prod_{i=1}^{r} T_i = \mathbb{I}\,.$$

Call a monodromy tupel **linearly rigid** if T is irreducible and up to conjugation in $\mathrm{GL}_n(\mathbb{C})$ uniquely determined by Jordan form of its elements.

Decomposition of linearly-rigid monodromy tupels

Example of CY-operator with linearly rigid monodromy

$$L^{(n+1)} \text{ for } {}_{n+1}F_n\left(\begin{matrix} a_1, \ \ldots \ , a_{n+1} \\ 1, \ldots , 1 \end{matrix} \,\middle|\, t\right) = P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & \\ \hline 0 & 0 & a_1 & \\ \vdots & & \vdots & t \\ 0 & n-1 & a_n & \\ 0 & n - \sum a_i & a_{n+1} & \end{array}\right\},$$

with certain $a_j \in \mathbb{Q} \backslash \mathbb{Z}$ for all j.

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with certain $a_j \in \mathbb{Q}\backslash\mathbb{Z}$ for all j.

Hadamard product on hypergeometric functions functions:

$$\sum_{n=0}^{\infty} A_n t^n \star_H \sum_{n=0}^{\infty} B_n t^n := \sum_{n=0}^{\infty} A_n B_n t^n ,$$

$${}_{n+1}F_n\left(\begin{matrix} a_1, a_2, \ldots, a_n, \alpha \\ c_1, \ldots c_{n-1}, 1 \end{matrix} \,\middle|\, t\right) \doteq \underbrace{{}_1F_0\left(\alpha; \,\middle|\, t\right)}_{= \frac{1}{(1-t)^{\alpha}}} \star_H {}_nF_{n-1}\left(\begin{matrix} a_1, a_2, \ldots, a_n \\ c_1, \ldots c_{n-1} \end{matrix} \,\middle|\, t\right)$$

(Similar for differential operators: **middle Hadamard product**.)

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with $a_j \in \mathbb{Q}\backslash\mathbb{Z}$ for all j.

Decomposition into rank-1 tupels:

$${}_{n+1}F_n\left(\begin{matrix} a_1, \ \dots \ , a_{n+1} \\ 1, \dots, 1 \end{matrix} \,\middle|\, t\right) = \underbrace{{}_1F_0(a_1; | t)}_{= \frac{1}{(1-t)^{a_1}}} \star_H \cdots \star_H {}_1F_0(a_{n+1}; | t) \,.$$

A similar procedure always works for any linearly-rigid monodromy.

Decomposition of linearly-rigid monodromy tupels

Decomposition into rank-1 tupels:

$$ {}_{n+1}F_n\left(\begin{array}{c} a_1, \ \dots \ , a_{n+1} \\ 1, \dots, 1 \end{array} \middle| \, t\right) = {}_1F_0(a_1; | t) \star_H \cdots \star_H {}_1F_0(a_{n+1}; | t) \, . $$

Proposition (N. Katz ['96])

If T is linearly rigid, it can be constructed via tensor- and middle Hadamard products of ${}_1F_0(\alpha; |.)$'s.

Deligne, N. Katz gave an arithmetic description of linear rigidity that generalizes to any reductive complex algebraic group.

Bogner, Reiter ['11] generalized decomposition result to Sp_4-rigid tupels.

But how ***geometric*** *are these decomposition results?*

Proposition (M.-Doran)

All rank-4 Calabi-Yau operators L of degree ≤ 2 and index ≤ 2 *are the Picard-Fuchs operators of one-parameter families of K3-fibered ($\rho = 18, 19$) EFS Calabi-Yau threefolds.*

There are 120 examples of this kind.

Proposition (M.-Doran)

All families are obtained through an **iterative construction** *that produces families of EFS Calabi-Yau n-folds from families of EFS Calabi-Yau varieties of one dimension lower using a* **generalized functional invariant**. *In particular, all families are iteratively constructed from a single geometric object, the deformed Fermat quadric given by*

$$X_0^2 + X_1^2 + 2tX_0X_1 = 0.$$

(Lian's period computations for surfaces of general type/Fano varieties have the same starting point.)

Calabi-Yau n-folds related by Hadamard twists

- Euler integral transform:

$$ {}_{n+1}F_n\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2} \\ 1,\ldots\ 1 \end{array}\middle|\, t\right) \doteq \int_0^1 \frac{dx}{\sqrt{x(1-x)}}\ {}_nF_{n-1}\left(\begin{array}{c} \frac{1}{2},\ldots,\frac{1}{2} \\ 1,\ldots\ 1 \end{array}\middle|\, t\,x\right) $$

- Hirarchy of twisted **Legendre pencils**:

$$ \begin{array}{ll} EC_t & y_1^2 = (1-t\,x_1)\,x_1\,(1-x_1)\,, \\ K3_t & y_2^2 = (1-t\,x_1\,x_2)\,x_1\,(1-x_1)\,x_2(1-x_2)\,, \\ CY3_t & y_3^2 = (1-t\,x_1\,x_2\,x_3)\,x_1\,(1-x_1)\,x_2(1-x_2)\,x_3\,(1-x_3)\,. \end{array} $$

- Compute their **periods**:

$$ \int_A \frac{dx_1}{y_1} = \int_0^1 \frac{dx_1}{\sqrt{x_1(1-x_1)}}\ \frac{1}{\sqrt{1-t\,x_1}} \doteq {}_2F_1\left(\frac{1}{2},\frac{1}{2};1\middle|\, t\right)\,, $$

$$ \iint_S \frac{dx_1\wedge dx_2}{y_2} = \int_0^1 \frac{dx_2}{\sqrt{x_2(1-x_2)}} \int_0^1 \frac{dx_1}{y_1} \doteq {}_3F_2\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,1 \end{array}\middle|\, t\right)\,, $$

$$ \iiint_C \frac{dx_1\wedge dx_2\wedge dx_3}{y_3} = \int_0^1 \frac{dx_3}{\sqrt{x_3(1-x_3)}} \iint_S \frac{dx_1\wedge dx_2}{y_2} \doteq {}_4F_3\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,1,1 \end{array}\middle|\, t\right)\,. $$

Calabi-Yau n-folds related by Hadamard twists

- Hirarchy of twisted **Legendre pencils**:

$$
\begin{array}{rlll}
pt_t & y_0^2 & = 1 - t & \text{(quadric pencil)}\,, \\
E_t & y_1^2 & = (1 - t\,x_1)\,x_1\,(1 - x_1) & \text{(extremal)}\,, \\
K3_t & y_2^2 & = (1 - t\,x_1\,x_2)\,x_1\,(1 - x_1)\,x_2(1 - x_2) & (\rho = 19)\,, \\
CY3_t & y_3^2 & = (1 - t\,x_1\,x_2\,x_3)\,x_1\,(1 - x_1)\,x_2(1 - x_2)\,x_3\,(1 - x_3) & (h^{2,1} = 1)\,.
\end{array}
$$

- Compute their **periods**:

$$
\begin{aligned}
\frac{1}{y_0} &= \frac{1}{\sqrt{1-t}} = {}_1F_0\left(\frac{1}{2};\middle|\, t\right)\,, \\
\int_A \frac{dx_1}{y_1} &= {}_1F_0\left(\frac{1}{2};\middle|\, t\right) \star_H {}_1F_0\left(\frac{1}{2};\middle|\, t\right) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1\middle|\, t\right)\,, \\
\iint_S \frac{dx_1 \wedge dx_2}{y_2} &= {}_1F_0\left(\frac{1}{2};\middle|\, t\right) \star_H {}_2F_1\left(\frac{1}{2},\frac{1}{2};1\middle|\, t\right) = {}_3F_2\left(\begin{array}{c}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{array}\middle|\, t\right)\,, \\
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\end{aligned}
$$

What about rk-2 rigid systems with 3 sing's?

- Rational elliptic surfaces $\mathbf{S}$

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2(t)\,x - g_3(t)\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{P}^1.$$

- Consider extremal families of elliptic curves with $\mathrm{rk}(\mathrm{MW}) = 0$, classified by Miranda, Persson ['86].
- Extremal rational surfaces (up to $*$-transfer w/ 3 sing.'s):

gen.	modular		μ	G
I_4	I_1	I_1^*	$1/2$	$\Gamma_0(4)$
I_3	I_1	IV^*	$1/3$	$\Gamma_0(3)$
I_2	I_1	III^*	$1/4$	$\Gamma_0(2)$
I_1	I_1	II^*	$1/6$	$\Gamma_0(1)^*$

Picard-Fuchs operators are rank-2 Calabi-Yau operators with holomorphic solution:

$$\omega = \oint_A \frac{dx}{y} = {}_2F_1(\mu, 1-\mu; 1|t)$$

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Picard-Fuchs operators are rank-2 Calabi-Yau operators with holomorphic solution:

$$\omega = \oint_A \frac{dx}{y} = \; {}_2F_1(\mu, 1-\mu; 1|t) = {}_2F_1\left(\begin{array}{c} \mu, 1-\mu \\ \frac{1}{2} \end{array} \middle| \, t \right) \star_H \; {}_1F_0\left(\tfrac{1}{2}; \middle| t\right)$$

Extremal rational surfaces and their periods

- **generalized functional invariant**: specifying integers (i, j, α) with $i \in \{1, 2\}$, $\alpha \in \{\frac{1}{2}, 1\}$, and $1 \leq j \leq 2\alpha$ such that

$$ {}_2F_1\left(\begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \middle| \, t \right) \doteq \oint_{|x|=\epsilon} \frac{dx}{x(x+1)^{\alpha}} \, {}_1F_0\left(\frac{1}{2}; \middle| \, \frac{c_{ij}\, t}{x^i (x+1)^j} \right) . $$

- min. **Weierstrass model** by twist *and* base transformation

$$ \begin{array}{lcl} pt_t : \quad y^2 & = & 1 - t \\ & \downarrow^{(i,j,\alpha)} & \\ EC_t : \quad y^2 & = & \left(1 - \frac{c_{ij}\, t}{x^i (x+1)^j}\right) x^2 (x-1)^{2\alpha} , \\ & & \Delta \doteq t^n (1-t) . \end{array} $$

with

(n, μ)	$(4, \frac{1}{2})$	$(3, \frac{1}{3})$	$(2, \frac{1}{4})$	$(1, \frac{1}{6})$
(i, j, α)	$(1, 1, 1)$	$(1, 2, 1)$	$(1, 1, \frac{1}{2})$	$(2, 1, \frac{1}{2})$

What about rk-3 rigid systems with 3 sing's?

- Twist EC_t with generalized functional invariant (i,j,α):

$$\begin{aligned} EC_t:\ y^2 &= 4\,x^3 - g_2(t)\,x - g_3(t) \\ &\big\downarrow {\scriptstyle (i,j,\alpha)} \\ K3_t:\ y^2 &= 4\,x^3 - g_2(T)\,h(u)^2 x - g_3(T)\,h(u)^3 \end{aligned}$$

- Base change: double cover $T = \frac{c_{ij}\,t}{u^i(u+1)^j}$, twist $h(u) = u^2(u+1)^{2\alpha}$.
- *Example* $\left(\mu = 1/4, (i,j,\alpha) = (1,1,1)\right)$: 1-param. family of K3 surfaces with M_2-polarization (Picard rank 19),

E_{sing}	I_2	I_1	III^*
t	0	1	∞

srfc. is rational

E_{sing}	I_4	$2\,I_1$	$2\,III^*$
s	∞	$T_t^{-1}(1)$	$0, -1$

srfc. is K3

- K3's are one-parameter families with $n = 1,2,3,4$ and $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ lattice polarization for $\mu = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.

One-parameter families of M_n-polarized K3 surfaces

- Twist EC_t with generalized functional invariant (i,j,α):

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- Base change: double cover $T = \frac{c_{ij}\,t}{u^i(u+1)^j}$, twist $h(u) = u^2(u+1)^{2\alpha}$.

Picard-Fuchs operators are rank-3 Calabi-Yau operators with holomorphic solution for $(i,j,\alpha) = (1,1,1)$:

$$\Omega = {}_3F_2\left(\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{array}\middle| t\right) = {}_1F_0\left(\tfrac{1}{2}; | t\right) \star_H {}_2F_1\left(\mu, 1-\mu; 1 | t\right)$$

- RHS can be interpreted as modular form for $\Gamma_0(n)^+$ with $n = 1, 2, 3, 4$ for $\mu = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.

What about rk-4 rigid systems with 3 sing's?

Example: *iterated* quadratic twists of extremal families of EC's

- Twist EC_t $(g_2(t), g_3(t)$ from modular surface for $\Gamma_0(n))$:

$$
\begin{array}{rcl}
EC_t:\; y^2 & = & 4\,x^3 - g_2(t)\,x - g_3(t) \\
 & \downarrow & \\
K3_t:\; y^2 & = & 4\,x^3 - g_2(tu)\,\big(u(1-u)\big)^2 x - g_3(tu)\,\big(u(1-u)\big)^3 \\
 & \downarrow & \\
CY3_t:\; y^2 & = & 4\,x^3 - g_2(tuv)\,\big(uv(1-u)(1-v)\big)^2 x - g_3(tuv)\,\big(uv(1-u)(1-v)\big)^3
\end{array}
$$

- Homogeneous Weierstrass equation over $\mathbb{P}^1 \times \mathbb{P}^1$:

$$Y^2\,Z = 4\,X^3 - G_2(t, U_0, U_1, V_0, V_1)\,X\,Z^2 - G_3(t, U_0, U_1, V_0, V_1)\,Z^3$$

  with $(\mathbb{C}^*)^3$-action (satisfying $\sum \deg(\mathrm{var}_i) = \deg(\mathrm{WEq})$).

- Canonical bundle formula for total space $\Rightarrow$ total space is CY.

- **Calabi-Yau threefold:**
  $= \Big((\mathrm{WEq}) - \{X = Y = Z = 0\} - \{U_1 = U_2 = 0\} - \{V_1 = V_2 = 0\}\Big) / (\mathbb{C}^*)^3$

What about rk-4 rigid systems with 3 sing's?

Example: *iterated* quadratic twists of extremal families of EC's

- Twist EC_t $(g_2(t), g_3(t))$ from modular surface for $\Gamma_0(n)$):

$$\begin{array}{rcl} EC_t : \; y^2 & = & 4\,x^3 - g_2(t)\,x - g_3(t) \\ & \downarrow_{(1,1,1)} & \\ K3_t : \; y^2 & = & 4\,x^3 - g_2(tu)\,\big(u(1-u)\big)^2 x - g_3(tu)\,\big(u(1-u)\big)^3 \\ & \downarrow_{(1,1,1)} & \\ CY3_t : \; y^2 & = & 4\,x^3 - g_2(tuv)\,\big(uv(1-u)(1-v)\big)^2 x - g_3(tuv)\,\big(uv(1-u)(1-v)\big)^3 \end{array}$$

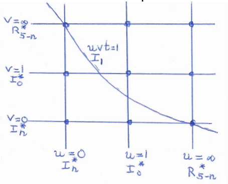
- 4 examples of Weierstrass models $W \to S$ over $S = \mathbb{P}^1 \times \mathbb{P}^1$

Picard-Fuchs operators are rank-4 Calabi-Yau operators with holomorphic solution:

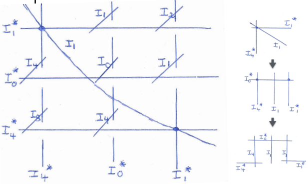
$$ {}_4F_3\left(\begin{array}{c} \mu, \frac{1}{2}, \frac{1}{2}, 1-\mu \\ 1, 1, 1 \end{array} \middle| t \right) = {}_1F_0\left(\tfrac{1}{2}; \middle| t\right) \star {}_1F_0\left(\tfrac{1}{2}; \middle| t\right) \star {}_2F_1\left(\begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \middle| t \right)$$

Smooth models for Weierstrass models

- following Miranda ['83] we obtain **smooth models** from the Weierstrass models $W \to S$ as follows:
  - blow-up S until discriminant Δ_{red} has simple normal crossings,
  - continue until only small list of collisions of Kodaira-types left,
  - obtain birational Weierstrass model $W' \to S'$,
  - <u>observe:</u> only blowing up $(4, 6, 12)$-points, so W' is still CY,
  - X', S' smooth; $f : X' \to W' \to S'$ flat, relat. minimal,
  - resolution of collision $II + IV$ must be handled separately.
- for $t \neq 0, 1$: $\left[EC_t : I_n, I_1, II^*/III^*/IV^* I_1^* \right]$



$\overset{n=4}{\Longrightarrow}$



- $h^{1,1}(S') = T + 1 = 2 + 31/21/15/13$ for $n = 1/2/3/4$

Elliptic fibrations on mirror families

- 1-parameter family of **Fermat hypersurface** in $\mathbb{P}^n$,

$$X_0^{n+1} + X_1^{n+1} + \cdots + X_n^{n+1} + (n+1)\,\lambda\, X_0 X_1 \cdots X_n = 0\,,$$

divide by $(\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$ to construct **mirror family**:

$$f_n(x_1,\ldots,x_n,t) = x_1 \cdots x_n \left(x_1 + \cdots + x_n + 1\right) + \frac{t}{(n+1)^{n+1}} = 0\,.$$

- Periods follow from **residue computation**:

$$\underbrace{\int \!\! \cdot\! \cdot \!\! \int_{K_n}}_{n-1} \frac{dx_2 \wedge \ldots dx_n}{\partial_{x_1} f_n(x_1,\ldots,x_n,t)} \doteq {}_nF_{n-1}\left(\begin{array}{ccc} \frac{1}{n+1} & \cdots & \frac{n}{n+1} \\ 1, & \ldots & ,1 \end{array} \middle|\, t \right)\,.$$

- *Iterative* structure, e.g., mirror-quartic $(n=3)$ is fibered by cubics $(n=2)$, by setting $\tilde{x}_1 = \frac{x_1}{x_3+1}, \tilde{x}_2 = \frac{x_2}{x_3+1}, \tilde{t} = -\frac{27\,t}{256\,x_3\,(x_3+1)^3}$.

Elliptic fibrations on mirror families

- families of EFS Calabi-Yau n-folds over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$,
- fibrations generated using **generalized functional invariant**,
- CY-operator and holomorphic solution by Hadamard-twist.

n	mirror	fiber	construction	period
1	quadric	2 points	–	${}_1F_0\left(\tfrac{1}{2};\,\middle|\, t\right)$
	WEq.: $Y^2 = 1 - t$			
2	cubic	EC's form rational srfc. with sing's I_3, I_1, IV^* and $MW = \mathbb{Z}/3\mathbb{Z}$	$(2,1,1)$	${}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\,\middle|\, t\right) \doteq {}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\\frac{1}{2}\end{matrix}\,\middle|\, t\right) \star {}_1F_0\left(\tfrac{1}{2};\,\middle|\, t\right)$
	WEq.: $y^2 = \left(1 - \frac{2^2\,t}{3^3 x^2\,(x+1)}\right)\,(x(x+1))^2 \quad \Rightarrow \quad Y^2 = 4X^3 - g_2(t)\,X - g_3(t)$			
3	quartic	K3's with M_2-pol., with sing's $I_{12}, 4I_1, IV^*$ and $MW = \mathbb{Z}/3\mathbb{Z}$	$\begin{cases}(3,1,1)\\(2,1,1)\end{cases}$	${}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{2}{4},\frac{3}{4}\\1,1\end{matrix}\,\middle|\, t\right) \doteq {}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{2}{4},\frac{3}{4}\\\frac{1}{3},\frac{2}{3}\end{matrix}\,\middle|\, t\right) \star {}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\,\middle|\, t\right)$
	WEq.: $Y^2 = 4X^3 - g_2\left(-\frac{3^3\,t}{4^4\,u^3\,(u+1)}\right)\,(u(u+1))^4\,X - g_3\,(\dots)\,\,(u(u+1))^6$			
4	quintic	CY3's over $\mathbb{P}^1 \times \mathbb{P}^1$ with $h^{2,1} = 1$, $h^{1,1} = 101$, fibered by M_3-pol. K3's	$\begin{cases}(3,2,1)\\(1,1,1)\\(2,1,1)\end{cases}$	${}_4F_3\left(\begin{matrix}\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\\1,1,1\end{matrix}\,\middle|\, t\right) \doteq {}_4F_3\left(\begin{matrix}\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\\\frac{1}{3},\frac{2}{3},\frac{1}{2}\end{matrix}\,\middle|\, t\right) \star {}_1F_0\left(\tfrac{1}{2};\middle|\, t\right) \star {}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\,\middle|\, t\right)$
	WEq.: $Y^2 = 4X^3 - g_2\left(\frac{3^3\,t}{2^2\,5^5\,v^3\,(v+1)^2\,u\,(u+1)}\right)\,(u(u+1)v(v+1))^4\,X - g_3\,(\dots)\,\,(u(u+1)v(v+1))^6$			

Summary

- Iterative procedure constructs families of EFS Calabi-Yau (n+1)-folds with $h^{n,1} = 1$, holom. periods, and PF operators:

$$\begin{array}{ccccccccc} pt_t & \stackrel{(i,j,\alpha)}{\longrightarrow} & EC_t & \stackrel{(1,1,1)}{\longrightarrow} & K3_t & \stackrel{(i',j',\alpha')}{\longrightarrow} & CY3_t & \stackrel{(1,1,1)}{\longrightarrow} & CY4_t \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}_1F_0 & \stackrel{\star}{\longrightarrow} & {}_2F_1 & \stackrel{\star}{\longrightarrow} & {}_3F_2 & \stackrel{\star}{\longrightarrow} & {}_4F_3 & \stackrel{\star}{\longrightarrow} & {}_5F_4 \end{array}$$

- allows for construction of Miranda elliptic fibrations,
- contains all fibrations of the Calabi-Yau threefolds from Doran, Morgan ['06] by M or M_n polarized K3's,
- produces all know examples of degree-2, rank-4 Calabi-Yau operators as geometric Picard-Fuchs operators.

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Thank your for your attention.