Mathematical Aspects of F-theory: A Status Report

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F-theory at 20

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Review: What is F-theory?

F-theory is a ten-dimensional quantum theory of gravity semi-classically approximated by:

- $SL(2,\mathbb{Z})$-equivariant type IIB SUGRA with quantized p-form fields, coupled to gauge theories along 7-branes and 3-branes with additional massless fields associated to brane intersections.

Mild singularities are allowed in the background spacetime which may introduce additional degrees of freedom.
The 7-branes are sources for a scalar field in the theory, and there is a richer variety of F-theory 7-branes than of string theory 7-branes.
One task for the future is to expand this formalism to allow for background 1-branes and 5-branes, as well as anti-branes.

Background fields in F-theory

The background fields in F-theory consist of

- An $\mathrm{SL}(2,\mathbb{Z})$-invariant complex scalar, specified by means of a complex line bundle $\mathcal{L}$ on spacetime, together with sections $f \in H^0(\mathcal{L}^{\otimes 4}), g \in H^0(\mathcal{L}^{\otimes 6})$.
- An $\mathrm{SL}(2,\mathbb{Z})$-doublet of 2-forms, whose expectation value is specified by a section of the sheaf $\mathcal{C}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{A}^2$, where $\mathcal{C}_{\mathbb{Z}}$ is the sheaf of possible integration contours for the elliptic integral $\int_\gamma \frac{dx}{\sqrt{x^3+fx+g}}$, i.e., the sheaf of local string charges.
- A self-dual 4-form field.
- 7-branes located at $\{4f^3 + 27g^2 = 0\}$, per a catalog. . .
- The field content of the 8D and 4D coupled theories (gauge fields and scalars), including Higgs vevs.
- Massless fields at brane intersections.
- A metric (with specified asymptotics near branes).
- Fermions and a gravitino.

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A catalog of 7-branes

type	$SL(2,\mathbb{Z})$ class	λ	monodromy	gauge algebra
$I_n,\, n \geq 1$	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	0	none	$\mathfrak{su}(n)$
			$\mathbb{Z}_2$	$\mathfrak{sp}([\frac{n}{2}])$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{1}{6}$	–	–
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{1}{4}$	–	$\mathfrak{su}(2)$
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\frac{1}{3}$	none	$\mathfrak{su}(3)$
			$\mathbb{Z}_2$	$\mathfrak{sp}(1)$
I_0^*	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\frac{1}{2}$	none	$\mathfrak{so}(8)$
			$\mathbb{Z}_2$	$\mathfrak{so}(7)$
			$\mathbb{Z}_3$ or $\mathfrak{S}_3$	$\mathfrak{g}_2$
$I_n^*,\, n \geq 1$	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	$\frac{1}{2}$	none	$\mathfrak{so}(2n+8)$
			$\mathbb{Z}_2$	$\mathfrak{so}(2n+7)$
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\frac{2}{3}$	none	$\mathfrak{e}_6$
			$\mathbb{Z}_2$	$\mathfrak{f}_4$
III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\frac{3}{4}$	–	$\mathfrak{e}_7$
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\frac{5}{6}$	–	$\mathfrak{e}_8$

Determining the gauge algebra

type	$\mathrm{ord}_{z=0}(f,g,\Delta)$	eqn. of monodromy cover
I_0	$(\geq 0, \geq 0, 0)$	–
I_1	$(0,0,1)$	–
I_2	$(0,0,2)$	–
$I_m,\ m\geq 3$	$(0,0,m)$	$\psi^2 + (9g/2f)\|_{z=0}$
II	$(\geq 1, 1, 2)$	–
III	$(1, \geq 2, 3)$	–
IV	$(\geq 2, 2, 4)$	$\psi^2 - (g/z^2)\|_{z=0}$
I_0^*	$(\geq 2, \geq 3, 6)$	$\psi^3 + (f/z^2)\|_{z=0}\cdot\psi + (g/z^3)\|_{z=0}$
$I_{2n-5}^*,\ n\geq 3$	$(2,3,2n{+}1)$	$\psi^2 + \frac{1}{4}(\Delta/z^{2n+1})(2zf/9g)^3\|_{z=0}$
$I_{2n-4}^*,\ n\geq 3$	$(2,3,2n{+}2)$	$\psi^2 + (\Delta/z^{2n+2})(2zf/9g)^2\|_{z=0}$
IV^*	$(\geq 3, 4, 8)$	$\psi^2 - (g/z^4)\|_{z=0}$
III^*	$(3, \geq 5, 9)$	–
II^*	$(\geq 4, 5, 10)$	–
non-min.	$(\geq 4, \geq 6, \geq 12)$	–

Explicit determination of the Kodiara type and monodromy (and hence the gauge algebra) for Weierstrass models.

The bundle

The line bundle captures an invariance in this setup: for a non-vanishing function u, replacing f and g by $u^4 f$ and $u^6 g$ does not change τ. The line bundle allows this ambiguity to be accounted for locally.

In addition, it may happen that $f = u^4 f'$, $g = u^6 g'$ for some bundle $\mathcal{W}$ and section u of that bundle (with $\mathcal{L} = \mathcal{W} \otimes \mathcal{L}'$).

In this case, τ is still identical between the two models (with a change of bundle).

A model for which there is no such u with a nontrivial $\mathcal{W}$ is known as a "minimal Weierstrass model." It is a result from mathematics that a minimal Weierstrass model always exists.

Dualities

F-theory is related to other quantum gravity theories by means of dualities.

- F-theory is the limit of M-theory when the spacetime is fibered by T^2's whose area goes to zero in the limit. From the F-theory perspective, the vacuum contains a circle whose radius goes to ∞ in the limit, leaving behind a 10D theory compactified on the base of the elliptic fibration.
  The typical spacetime here is a resolution of the total space of a Weierstrass fibration (or of an elliptic Calabi–Yau torsor).
- F-theory vacua which are fibered by S^2's may have a limit in which the S^2's break in half; in this case, there is a duality with certain vacua of the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$-heterotic string. There is also an $\mathfrak{so}(32)$-heterotic variant of this.
- F-theory vacua may have a limit which is pure type IIB string theory including D7-branes and orientifold planes (the "Sen limit").

1. The frozen phase of F-theory

- There are known to be M-theory vacua with ADE singularities which have no associated enhanced gauge symmetry. These singularities also feature a nonzero value for the integral of the M-theory 3-form field over the boundary of a neighborhood of the singularity.
- Tachikawa argued in arXiv:1508.06679 that precisely one of these "frozen" singularities lifts to F-theory, namely, the singularity which describes an $O7^+$ orientifold plane in type IIB string theory. Witten had argued in hep-th/9712028 that the monodromy around such an $O7^+$-plane is $\begin{bmatrix} -1 & -4 \\ 0 & -1 \end{bmatrix}$, the same as for an I_4^* Kodaira fiber. More generally one can put additional $D7$-branes on top of the orientifold plane yielding a monodromy of the form $\begin{bmatrix} -1 & -4-k \\ 0 & -1 \end{bmatrix}$ and a symplectic (rather than orthogonal) gauge group.

1. The frozen phase of F-theory

- In work in progress with Bhardwaj, Tachikawa, and Tomasiello, we are working out the details of this "frozen phase" of F-theory. For certain (perhaps all) Weierstrass models with one or more singular fibers of type I^*_{4+k}, there is an alternate compactification of F-theory with some discrete flux turned on, with a different dictionary than the usual one for the gauge algebra and matter content.
- Somewhat surprisingly, for 6D models the anomaly cancellation – although highly constrained – does admit solutions of this type, at least for the gauge and mixed anomalies. Calculating the purely gravitational anomaly is a challenge since the discrete flux influences the calculation.

2. Limitations of Tate's algorithm

- Tate's algorithm was originally developed in number theory to refine Kodaira's original procedure for determining the type of singular fiber in a Weierstrass model.
- Although Tate's algorithm was originally designed to detect things happening in codimension one on the base (i.e., gauge symmetries from a physical point of view), in hep-th/9605200 and arXiv:1106.3854 the algorithm was used to study phenomena in codimension two under some assumptions about the singularities present on the discriminant locus.
- Even with these assumptions, the analysis in arXiv:1106.3854 is known to be incomplete: we have no general description of the form of the equation in cases $SU(m)$ with $6 \leq m \leq 9$, $Sp(n)$ with $n = 3, 4$, or $SO(\ell)$ with $\ell = 13, 14$. To fully understand the codimension two phenomena in these cases, we need to complete the Tate-algorthm-inspired analysis.

3. The matter content of an F-theory vacuum

- More generally, we need an expanded dictionary for possible matter content in F-theory models. The Tate's algorithm analysis (and the Katz–Vafa analysis to which it is closely related) provides a wide array of matter representation but there are some representations which are more subtle to explain.
- In particular, when the local ring of the discriminant locus at a particular point is not a unique factorization domain, the Tate's algorithm analysis breaks down completely. There are now many interesting examples known with more exotic matter (work of Taylor and various collaborators) but a systematic procedure for determining the matter content is not known.

3. The matter content of an F-theory vacuum

- From a mathematical perspective, one would like to know the structure of the singular fibers along any locus of codimension two in the base, once the singularities in the Weierstrass model have been resolved. One would also like to know the intersection numbers of the components of the singular fiber with the divisors in the resolved Weirstrass model, which will determine the charges present in the representation. Finally – and this is the most subtle point – when the divisors themselves have some monodromy, one would like to determine the corresponding representation content. (It is not clear whether or not this latter step is a purely mathematical question.)

4. Log terminal singularities and birational geometry

- When B has dimension 1, Kodaira proved a formula for the canonical bundle of the total space of the elliptic fibration $\pi : X \to B$, which for simplicity we state in case there are no multiple fibers. Kodaira showed that $K_X = \pi^*(K_B + \Lambda)$, where $\Lambda = \sum \lambda_i D_i + \frac{1}{12} j^*[P]$. Here, λ_i are the coefficients associated to the various Kodaira types in the catalog, and P represents some fixed point on the j-line, being pulled back to a divisor on B by the j-function.
- Note that Λ is a $\mathbb{Q}$-divisor on B, and this formula really gives a formula for $12K_X$ with integer coefficients.
- Kodaira's canonical bundle formula has been extended in various ways to higher dimension. Let us say that $\pi : X \to B$ is a good elliptic fibration if $K_X = \pi^*(K_B + \Lambda)$.

4. Log terminal singularities and birational geometry

- In higher dimension, the singularities of B and the singularities of Λ become relevant. Algebraic geometers, motivated by the study of birational geometry (blowing up and blowing down) have introduced the notion of a "klt pair" (B, Λ). There are two properties: (1) all coefficients of Λ must be strictly less than 1, and (2) there is a "log resolution" of singularities $f : (\widetilde{B}, \widetilde{\Lambda}) \to (B, \Lambda)$ such that the difference

$$K_{\widetilde{B}} + \widetilde{\Lambda} - f^*(K_B + \Lambda)$$

  is an effective divisor.
- Example: if B has dimension 2 and Λ is empty, this is equivalent to $B = \mathbb{C}^2/\Gamma$ for some $\Gamma \subset U(2)$ with the origin as isolated fixed point.
- "Minimal model" property: $(K_B + \Lambda) \cdot C \geq 0$ for all curves C on B.

4. Log terminal singularities and birational geometry

- The work of Grassi from the early 1990's shows that for an elliptic Calabi–Yau threefold:
  1. First we can blow up base and fiber so that both are nonsingular and that the family $\pi : X \to B$ is flat (i.e., all fibers have dimension one) and "good."
  2. Then, we can blow back down, achieving the key "minimal model" property $K_X \cdot C \geq 0$ for all C on the total space, and simultaneously achieving the "log minimal model" property $(K_B + \Lambda) \cdot C \geq 0$ for all C on the base. The family is still flat and "good."
  3. The singularities of the minimal model of X are at worst $\mathbb{Q}$-factorial terminal singularities, while the singularities of the minimal model of (B, Λ) are at worst klt.
- Unpublished work of Grassi extends much of this picture to the case of elliptic Calabi–Yau fourfolds.

5. Calabi–Yau varieties with terminal singularities

- The singularities of X are called canonical if there is a resolution of singularities $f : Y \to X$ such that $K_Y - f^*(K_X)$ is an effective divisor. The singularities are called terminal if in addition every exceptional divisor of the blowup map occurs in $K_Y - f^*(K_X)$ with a nonzero coefficient. The singularities are called $\mathbb{Q}$-factorial if an integer multiple of every codimension one subvariety can be defined by a single equation (i.e., every Weil divisor is $\mathbb{Q}$-Cartier).
- As mentioned earlier, these conditions on singularities are the natural outcome of studying blowing down on algebraic varieties: the endpoint of blowing down (and flips) leaves one with an algebraic variety having at worst $\mathbb{Q}$-factorial terminal singularities.
- Weierstrass models have canonical singularities, and some blowing up will be needed to get them into "$\mathbb{Q}$-factorial terminal" form.

6. The abelian part of the gauge group

- We need a better understanding of the abelian part of the gauge group. On a resolution X of the Weierstrass model, the abelian part is described in terms of the Cartier divisors on X, but those divisors may fail to be $\mathbb{Q}$-Cartier on the Weierstrass model itself. The presence of non-$\mathbb{Q}$-Cartier divisors suggests that at least some singular points will not be $\mathbb{Q}$-factorial. Should we more naturally consider this as a property of the local singularities, or as a more global property of the entire Weiestrass model?
- A related issue: from the perspective of the Weierstrass coefficients themselves, we do not have an algorithm for computing the Mordell–Weil group of $y^2 = x^3 + fx + g$.

7. The discrete part of the gauge group

- We need to understand the possible discrete symmetries more algorithmically. Mathematically, this boils down to the problem of finding all of the elliptic Calabi–Yau torsors for a given Weierstrass model.
- An elliptic Calabi–Yau torsor for a Weierstrass model $W \to B$ is a Calabi–Yau variety X which is fibered by curves of genus one $\pi : X \to B$ such that the elliptic curve $E_{W/B}$ over the function field of B acts on the genus one curve $C_{X/B}$:

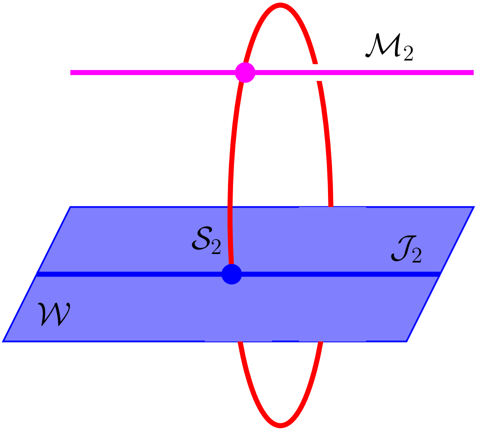
$$E_{W/B} \times C_{X/B} \to C_{X/B}.$$

- The reason this notion is important is that the τ functions of $W \to B$ and $X \to B$ are identical. Since F-theory is only sensitive to the τ function, compactifying M-theory on any of the torsors of $W \to B$ and taking the F-theory limit must lead to the same F-theory model.

7. The discrete part of the gauge group

- How can this happen physically? In other words, how can there be distinct models in M-theory with a common F-theory limit? We already understand this for continuous parameters (the Kaluza–Klein mechanism), but for discrete choices the lesson is: discrete choices of M-theory vacua correspond to discrete symmetries in F-theory.
- Let us recall how this works in a particular example (which I worked out with Taylor), in which the discrete symmetry in F-theory can also be seen to arise via the Higgs mechanism. Taylor and I started with a theory having a $U(1)$ gauge symmetry (from the Mordell–Weil group) and two kinds of charged matter. We found the deformation of complex structure which Higgsed the matter of charge 2, and this left us with a residual $\mathbb{Z}_2$ gauge symmetry.

7. The discrete part of the gauge group



7. The discrete part of the gauge group

- Just as we are missing a good algorithm for determining the Mordell–Weil group of an elliptic fibration, we are missing a good algorithm for determining the set of elliptic Calabi–Yau torsors. In work in progress, I am attempting to find such an algorithm.

8. Intermediate Jacobians and T-branes

- We need an improved understanding of the role of intermediate Jacobians and of T-branes. Anderson, Heckman, and Katz gave a beautiful geometric interpretation to the role played by the the 3-form field (intermediate Jacobian) in M-theory compactifications of F-theory models involving T-branes, particularly when the total space has singularities. They gave details for Calabi–Yau threefolds; the picture still needs to be worked out in detail for Calabi–Yau fourfolds.
- The M-theory 3-form field corresponds under the Kaluza–Klein analysis to the self-dual 4-form field in type IIB supergravity. It would be ideal to have an intrinsic way to specify this part of the data directly in terms of the self-dual 4-form. This would also be useful when discrete flux effects are present, as in the case of the frozen phase of F-theory.

9. Fluxes and Instantons

- We need a systematic way to understand fluxes and instantons in detail. The KKLT analysis and its variants in F-theory suggest that these are the key effects for stabilizing moduli. The fluxes and instantons are typically studied in M-theory rather than directly in F-theory; ideally, there would be a translation to direct F-theory properties which would then enable these fluxes to be analyzed mathematically.
- Let me advertise some work concerning fluxes in M-theory (joint with Jockers, Katz, and Plesser) which Katz will talk about on Thursday.

10. Finiteness

Is the set of complete families of elliptically fibered Calabi–Yau fourfolds finite (up to birational automorphism)? Taylor and collaborators are making significant progress in surveying and classifying these models.

Summary: Mathematical Aspects

Here are the mathematical aspects we have focussed on.

- We began with a review of F-theory. Primary interpretation in terms of IIB with variable dilaton (i.e., $\mathrm{SL}(2, \mathbb{Z})$-equivariant type IIB string theory), but many important properties are studied either via duality to M-theory after a circle compactification, duality to the heterotic string after a degeneration of complex structure, or duality to IIB orientifold models after a different degeneration of complex structure.

1. The frozen phase of F-theory.

Summary: Mathematical Aspects

2. We need to complete Tate's algorithm (or better understand the situation) for certain classical groups.
3. We need an improved dictionary for determining the matter content of an F-theory model. One aspect of this is the fact that there is some ambiguity between matter content at codimension one and at codimension two (in the base). Another aspect is that when the discriminant locus has complicated singularities, the standard approach (derived from Tate's algorithm which was originally intended to answer questions about the number theory of elliptic curves) is inadequate for anwering the question.

Summary: Mathematical Aspects

4. Motivated by Kodaira's canonical bundle formula on the one hand and birational geometry of the base of an elliptic fibration on the other hand, we introduced pairs (B, Λ) (with the coefficients of Λ determined by the Kodaira classification) as well as the notion of "klt pair" which restricts the singularity type.
   The birational geometry of F-theory models in dimension six is quite well understood, thanks to work of Grassi from the early 1990's, and has been extended in part to dimension four.

Summary: Mathematical Aspects

5. Birational geometry of threefolds and fourfolds: there are two natural classes of singularities, terminal singularities and canonical singularities, which do not affect the canonical divisor. (There is one related property of singularities: whether or not the singularity is "$\mathbb{Q}$-factorial".) The mathematics strongly suggests that such singular Calabi–Yau threefolds and fourfolds should be on an equal footing with the nonsingular ones. We saw several instances where the physical interpretation is clear and important.

Summary: Mathematical Aspects

6. We need a better understanding of the abelian part of the gauge group. On a resolution X of the Weierstrass model, the abelian part is described in terms of the Cartier divisors on X, but those divisors may fail to be $\mathbb{Q}$-Cartier on the Weierstrass model itself. The presence of non-$\mathbb{Q}$-Cartier divisors suggests that at least some singular points will not be $\mathbb{Q}$-factorial. Should we more naturally consider this as a property of the local singularities, or as a more global property of the entire Weiestrass model?

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8. We need an improved understanding of the role of intermediate Jacobians and of T-branes.
9. We need a systematic way to understand fluxes and instantons in detail.
10. Is the set of complete families of elliptically fibered Calabi–Yau fourfolds finite (up to birational automorphism)?