Comments on K3 Moduli Space

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Dave Day, F-Theory at 20
Caltech, 25 February 2016

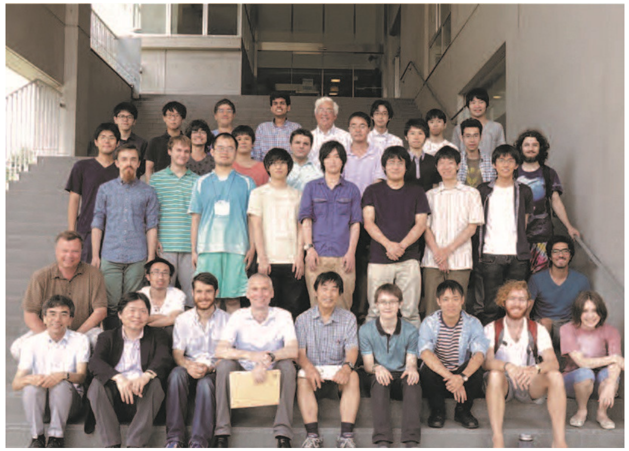
20 years ago in Kyoto

Many summers in Aspen

by Sav Sethi

At Kavli IPMU

Advisory Committee



Workshops

Fun with K_3

Freeman Dyson at the **Ramanujan Centenary Conference** in 1987:

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. ... My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions...

Eguchi - Taormina

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$$\mathrm{ch}_0^{\mathrm{R}}(l=0;\,-1/\tau)=\mathrm{ch}_0^{\mathrm{NS}'}(l=1/2;\,\tau)+\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\cosh\pi\alpha}\,\mathrm{ch}^{\mathrm{NS}'}(h=\alpha^2/2-1/8;\,\tau)\,,$$

$$\mathrm{ch}_0^{\mathrm{NS}}(l=1/2;\,-1/\tau)=-\mathrm{ch}_0^{\mathrm{N}}(l=1/2;\,\tau)+\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\cosh\pi\alpha}\,\mathrm{ch}^{\mathrm{NS}}(h=\alpha^2/2-1/8;\,\tau)\,.$$

$$\mathrm{ch}_0^{\mathrm{R}}(k=1,\,l=0;\,z)=\sum_m q^{m^2/2+m/2+1/4}z^{m+1/2}\frac{1}{1+zq^m}f^{\mathrm{R}}(z)\,,$$

$$\mathrm{ch}_0^{\mathrm{R}}(k=1,\,l=1/2;\,z)=\sum_m q^{m^2/2+m/2+1/4}z^{m+1/2}\frac{zq^m-1}{1+zq^m}f^{\mathrm{R}}(z)\,,$$

$$\mathrm{ch}_0^{\mathrm{NS}}(k=1,\,l=0;\,z)=\sum_m q^{m^2/2}z^m\frac{zq^{m-1/2}-1}{1+zq^{m-1/2}}f^{\mathrm{NS}}(z)\,,$$

$$\mathrm{ch}_0^{\mathrm{NS}}(k=1,\,l=1/2;\,z)=\sum_m q^{m^2/2}z^m\frac{1}{1+zq^{m-1/2}}f^{\mathrm{NS}}(z)\,,$$

Elliptic genus of K3 expanded in N=4 superconformal characters

$$F(\tau) \equiv \sum_h (N_{h,1} - 2N_{h,0}) q^h = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots$$

$$F(-1/\tau) = \sum_h (N_{h,1} - 2N_{h,0}) \cdot \tilde{q}^h \longrightarrow 2\sqrt{-i\tau q}^{-1/8} - 12 + \cdots .$$

This observation seems to imply that the q-expansion coefficients of $F(\tau)$ are all positive and the symmetry of the generic non-linear σ-model is just the $N = 4$ superconformal symmetry, though I have no rigorous proof for it. Of course this

My PhD thesis (1989)

Elliptic genus of K3 expanded in N=4 superconformal characters

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21 years later...

Representations of Mathieu group M24

n	1	2	3	4	5
A_n	45	231	770	2277	5796

Eguchi - Tachikawa - HO, arXiv: 1004.0956

K_3 moduli space

based on work in progress
with J. Gomis, Z. Komargodski,
N. Seiberg, and Y. Wang.

For 2d $\mathcal{N}=(2,2)$ SCFT,
we learn that the moduli space splits.

$$\mathcal{M} = \mathcal{M}_{cc} \times \mathcal{M}_{ca}$$

$$G_L^- G_R^- \chi \qquad\qquad G_L^- G_R^+ \phi$$

χ : (c,c) primary $\qquad$ ϕ : (c,a) primary

The moduli space of $K3$ sigma-model

is locally $O(4,20)/O(4)\times O(20)$.

conjectured by Seiberg (1988)

proven by Aspinwall-Morrison (1994)

- quaternion Kähler
- does not split
- not even Kähler

Why?

To my knowledge, the original proof of $\mathcal{M} = \mathcal{M}_{cc} \times \mathcal{M}_{ca}$ is by Dixon-Kaplunovsky-Louis (1989).

- hybrid of stringy S-matrix computation and supergravity analysis.

Geometry of Moduli Space

$$S \to S + \lambda^i \int d^2x\, \mathcal{O}_i(x)$$

OPE of truely marginal operators may have contact terms.

Seiberg (1988)

$$\mathcal{O}_i(x)\,\mathcal{O}_j(y) = \frac{g_{ij}}{|x-y|^4} + \Gamma_{ij}^{\ k}\,\mathcal{O}_k(y)\,\delta^2(x-y) + \cdots$$

$\Gamma_{ij}^{\ k}$: connection on the moduli space

Kutasov (1988)

To derive $\mathcal{M} = \mathcal{M}_{cc} \times \mathcal{M}_{ca}$ from the worldsheet perspective, we use the method developed by Ranganathan - Sonoda - Zwiebach (1993).

$R_{ij\,ab}$: curvature of local fields $\{\phi_a\}$

$$= \int_{|x|\leq 1} d^2x \int_{|y|\leq 1} d^2y \, \langle \mathcal{O}_i(x)\, \mathcal{O}_j(y)\, \phi_a(\infty)\, \phi_b(0) \rangle - (x \leftrightarrow y)$$

Since the integrand is anti-symmetric in $x \leftrightarrow y$,

$R_{ij\,ab} = 0$ without OPE singularities.

de Boer, et.al. (arXiv: 0809.0507) applied it to $\mathcal{N}=(2,2)$ theory to show:

$$R_{i\bar{j}k\bar{l}} = \int_{|x|\leq 1} d^2x \int_{|y|\leq 1} d^2y \, \langle G_L^- G_R^- \chi_i(x) \, G_L^+ G_R^+ \bar{\chi}_{\bar{j}}(y) \, \chi_k(\infty)\, \chi_{\bar{l}}(0) \rangle - (x \leftrightarrow y)$$

$$= [\, C_i , \bar{C}_{\bar{j}} \,]_{k\bar{l}} + \alpha \, g_{i\bar{j}} \, g_{k\bar{l}}$$

reproducing the tt^* equation.

de Boer et. al also computed the curvature of $\mathcal{N}=2$ currents $G_L^{\pm}$:

$$\int_{|x|\le 1} d^2x \int_{|y|\le 1} d^2y \ \langle G_L^- G_R^- \chi_i(x)\, G_L^+ G_R^+ \bar{\chi}_{\bar{j}}(y)\ G_L^+(\infty)\, G_L^-(0) \rangle \\ - (x \leftrightarrow y)$$

$\Rightarrow$ curature $\propto$ $G_{i\bar{j}}$: Zamolodchikov metric.

G_L^+ is an operator-valued section of $[\text{Hodge bundle}]^{3/c}$

How about (c, c) primaries χ_i over $\mathcal{M}_{ca}$?

$$\langle G_L^- G_R^+ \phi_a(x)\, G_L^+ G_R^- \bar{\phi}_{\bar{b}}(y)\, \chi_i(\infty)\, \bar{\chi}_{\bar{j}}(0) \rangle$$

$$= \partial_y \partial_{\bar{y}} \left(\frac{\bar{y}}{\bar{x}} \langle \phi_a(x)\, \bar{\phi}_{\bar{b}}(y)\, \chi_i(\infty)\, \bar{\chi}_{\bar{j}}(0) \rangle \right)$$

Ward identity.

$$R_{a\bar{b}i\bar{j}} = 0$$

if $\phi_a(x)\, \chi_i(0)$ is smooth

as $x \to 0$.

$R_{a\bar{b}i\bar{j}} = 0$ if $\phi_a(x)\chi_i(0)$ is smooth as $x \to 0$.

$$\Rightarrow \quad \mathcal{M} = \mathcal{M}_{cc} \times \mathcal{M}_{ca}$$

This condition is violated in theories with $\mathcal{N} = (4,4)$ supersymmetry.

$\mathcal{N} = (4,4)$ SCFT : 4 supercurrents $G_L^{\pm}$, $\tilde{G}_L^{\pm}$

(c,c) and (c,a) primaries are related by

$SU(2)_R$ symmetry. $\chi_i(0) \sim \oint dz\, J_R^{++}(z)\, \phi_i(0)$

The $SU(2)_R$ symmetry also generates

new OPE singularities :

$$\chi_i(z,\bar{z})\, \phi_j(w,\bar{w}) \sim \frac{\eta_{ij}}{\bar{z}-\bar{w}}\, J_L^{++}(w)$$

$$\left(\Phi_i{}^{\alpha\dot\alpha}\right)_{\substack{\alpha=1,2\\ \dot\alpha=\dot1,\dot2}} \equiv \begin{pmatrix} \chi_i & \phi_i \\ \bar\phi_i & \bar\chi_i \end{pmatrix}$$

$$\Phi_i{}^{\alpha\dot\alpha}(z,\bar z)\ \Phi_j{}^{\beta\dot\beta}(w,\bar w)$$

$$= \eta_{ij}\left(\frac{\epsilon^{\alpha\beta}}{z-w} + \sigma_a^{\alpha\beta}\, J_L^a(w)\right)$$

$$\times\left(\frac{\epsilon^{\dot\alpha\dot\beta}}{\bar z-\bar w} + \sigma_a^{\dot\alpha\dot\beta}\, J_R^a(\bar w)\right)$$

Using

$$\chi_i(z,\bar{z})\ \phi_j(w,\bar{w}) \sim \frac{\eta_{ij}}{\bar{z}-\bar{w}} J_L^{++}(w) \ ,$$

we find

$$\int_{|x|\leq 1} d^2x \int_{|y|\leq 1} d^2y \ \langle G_L^- G_R^+ \phi_i(x)\ G_L^+ G_R^- \bar{\phi}_j(y)\ \chi_k(\infty)\, \chi_l(0) \rangle$$

$$- (x \leftrightarrow y)$$

$$= \eta_{il}\, \eta_{jk} - \eta_{ik}\, \eta_{jl}$$

$$\int_{|x|\le 1} d^2x \int_{|y|\le 1} d^2y \, \langle G_L^- G_R^+ \phi_i(x) \, G_L^+ G_R^- \bar{\Phi}_j(y) \, \chi_k(\infty) \, \chi_l(0) \rangle$$

$$- \, (x \leftrightarrow y)$$

$$= \eta_{il} \, \eta_{jk} - \eta_{ik} \, \eta_{jl}$$

(c.c) primaries are fibered over $\mathcal{M}_{c.a}$.

$\Rightarrow$ $\mathcal{M}$ does not split as $\mathcal{M}_{c.c} \times \mathcal{M}_{ca}$.

We can also compute the curvature of $\mathcal{N}=4$ supercurrents over $\mathcal{M}$.

$$(\Phi_i{}^{\alpha\dot\alpha}) = \begin{pmatrix} \chi_i & \phi_i \\ \bar\phi_i & \bar\chi_i \end{pmatrix}, \quad (G^{\alpha A}) = \begin{pmatrix} G^+ & \tilde G^- \\ \tilde G^+ & G^- \end{pmatrix}$$

$\alpha = 1,2$: $SU(2)_R$ inner automorphism

$A = 1.2$: $SU(2)$ outer automorphism

$$R_{(i:A\dot A)(j:B\dot B)(\gamma,C)(\delta,D)} \leftarrow G^{\gamma,C}\, G^{\delta D}$$

$$= -2\,\eta_{ij}\,\epsilon_{\dot A\dot B}\,\epsilon_{\gamma\delta}\left(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}\right)$$

$$R_{(i:A\dot{A})(j:B\dot{B})(\gamma,C)(\delta,D)} \leftarrow G^{\gamma,C}\ G^{\delta D}$$

$$= -2\,\eta_{ij}\,\epsilon_{\dot{A}\dot{B}}\,\epsilon_{\gamma\delta}\,(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})$$

This means :

- $G^{\pm}$ and $\tilde{G}^{\pm}$ don't mix if we stay on $\mathcal{M}_{c,c}$ (or $\mathcal{M}_{c,a}$)
- G^{+} and $\tilde{G}^{+}$ (G^{-} and $\tilde{G}^{-}$) mix if we move around a generic loop in the full moduli space.

Summary:

The curvature is generated because of the need to regularize.
Otherwise, the x, y - integrals vanish.

N=2 superconformal Ward indentities imply that the moduli space splits
if there are no OPE singularities between (c,c) and (c,a) marginal primaries.

This condition is violated in N=4 SCFT.
This also happens in other cases with extende supersymmetry, e.g., tori.

N=4 supercurrents have non-Abelian curvature over the moduli space.

$\Rightarrow$ N=2 subalgebra cannot be chosen consistently.

Happy Birthday, Dave!