Matrix	# of Them
$1_{4\times 4}$	1
γ^μ	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	6
$\gamma^\mu \gamma^5$	4
γ^5	1

Any 4x4 matrix Γ^A is a linear combination of these

Note $P_{R,L} = \left(\frac{1 \pm \gamma_5}{2}\right)$ are projection operators onto different subspaces

$P_L + P_R = 1$, $P_R P_L = P_L P_R = 0$, $P_R^2 = P_R$, $P_L^2 = P_L$

The matrices in the above basis either commute or anticommute with γ^5

$[\Gamma^I, \gamma_5] = 0$ $\quad 1, \sigma^{\mu\nu}, \gamma^5 \quad P_L \Gamma^I P_R = P_R \Gamma^I P_L = 0$

$\{\Gamma^I, \gamma_5\} = 0$ $\quad \gamma^\mu, \gamma^\mu\gamma^5 \quad P_L \Gamma^I P_L = P_R \Gamma^I P_R = 0$

Matrices Γ^I in our basis are linearly independent + span space of 4x4 matrices. Can write, for example

$$\gamma^\alpha \gamma^\beta \gamma^\delta = \sum_I C_I \Gamma^I$$

Let's work out this expansion explicitly

First

$\{\gamma^\alpha\gamma^\beta\gamma^\delta, \gamma^5\} = 0$

So $\gamma^\alpha\gamma^\beta\gamma^\delta$ is a lc of γ^μ, $\gamma^\mu\gamma^5$. Also

$$D(\Lambda)^{-1}\gamma^\alpha\gamma^\beta\gamma^\delta D(\Lambda) = D(\Lambda)^{-1}\gamma^\alpha D(\Lambda)\, D(\Lambda)^{-1}\gamma^\beta D(\Lambda)\, D(\Lambda)^{-1}\gamma^\delta D(\Lambda)$$
$$= \Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}\Lambda^\delta{}_{\delta'}\,\gamma^{\alpha'}\gamma^{\beta'}\gamma^{\delta'}$$

So must "transform" as a rank 3 tensor

$$\gamma^\alpha\gamma^\beta\gamma^\delta = X\eta^{\alpha\beta}\gamma^\delta + Y\eta^{\beta\delta}\gamma^\alpha + Z\eta^{\alpha\delta}\gamma^\beta$$
$$+ W i\epsilon^{\alpha\beta\delta\eta}\gamma_\eta\gamma_5 + \cdots$$

More terms with $\gamma^\alpha\gamma_5$ & no epsilon & $\epsilon\gamma_\eta$ with no γ_5. In fact we can argue these don't occur. Recall under parity

$$\psi \to \pm\gamma^0\psi \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^0\gamma^i\gamma^j\gamma^k\gamma^0 = -\gamma^i\gamma^j\gamma^k$$
$$\gamma^0\eta^{ij}\gamma^k\gamma^0 = +\eta^{ij}\gamma^k\gamma^5$$

$\leftarrow$ wrong sign

$$\gamma^0\epsilon^{ijkm}\gamma_m\gamma^0 = \gamma^0\epsilon^{ijk0}\gamma_0\gamma_0 = +\epsilon^{ijkm}\gamma_m$$

So Ellipses vanish & we have

$$\gamma^\alpha\gamma^\beta\gamma^\delta = X\eta^{\alpha\beta}\gamma^\delta + Y\eta^{\beta\delta}\gamma^\alpha + Z\eta^{\alpha\delta}\gamma^\beta + W i\epsilon^{\alpha\beta\delta\eta}\gamma_\eta\gamma_5$$

Symmetrize on α and β: $(\alpha\beta + \beta\alpha)/2$

$$\eta^{\alpha\beta}\gamma^\delta = X\eta^{\alpha\beta}\gamma^\delta + \frac{Y}{2}\left(\eta^{\beta\delta}\gamma^\alpha + \eta^{\alpha\delta}\gamma^\beta\right)$$

$$+ \frac{Z}{2}\left(\eta^{\alpha\delta}\gamma^\beta + \eta^{\beta\delta}\gamma^\alpha\right) + 0$$

$$\Rightarrow \quad X = 1 \quad , \quad Y + Z = 0$$

Next symmetrize on β and δ

$$\gamma^\alpha\eta^{\beta\delta} = \frac{X}{2}\left(\eta^{\alpha\beta}\gamma^\delta + \eta^{\alpha\delta}\gamma^\beta\right)$$

$$+ Y\eta^{\beta\delta}\gamma^\alpha + \frac{Z}{2}\left(\eta^{\alpha\delta}\gamma^\beta + \eta^{\alpha\beta}\gamma^\delta\right) + 0$$

$$\Rightarrow \quad X + Z = 0 \quad , \quad Y = 1$$

So we have

$$X = Y = 1 \, , \quad Z = -1$$

It only remains to determine W. Pick $\alpha = 0$, $\beta = 1$, $\gamma = 2$

$$\gamma^0\gamma^1\gamma^2 = W i \overset{=1}{\epsilon^{0123}} \gamma_3\gamma_5$$

$$\gamma^0\gamma^1\gamma^2\gamma^3 = W i \gamma_3\gamma_5\gamma^3 = -W i \gamma_5$$

$$\Downarrow$$

$$-i\gamma_5 = -W i \gamma_5 \quad \Rightarrow \quad W = 1$$

$$\gamma^\alpha\gamma^\beta\gamma^\delta = \eta^{\alpha\beta}\gamma^\delta + \eta^{\beta\delta}\gamma^\alpha - \eta^{\alpha\delta}\gamma^\beta + i\epsilon^{\alpha\beta\delta\eta}\gamma_\eta\gamma_5$$

There are several other useful identities

$$\mathrm{Tr}\,\gamma^\alpha\gamma^\beta = \mathrm{Tr}\,\tfrac{1}{2}(\gamma^\alpha\gamma^\beta+\gamma^\beta\gamma^\alpha) = \mathrm{Tr}\,\eta^{\alpha\beta}\mathbb{1} = 4\eta^{\alpha\beta}$$

Trace of an odd # of Gamma matrices vanish

$$\mathrm{Tr}\,\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}} = \mathrm{Tr}\,\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma_5\gamma_5$$

$$= -\mathrm{Tr}\,\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}\gamma_5\gamma^{\mu_{2n+1}}\gamma_5$$

$$= (-1)^{2n+1}\,\mathrm{Tr}\,\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma_5$$

$$= -\mathrm{Tr}\,\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}$$ move over proper $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$

$$\Rightarrow \mathrm{Tr}\,\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}} = 0$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 2\eta^{\alpha\beta}\mathrm{Tr}\,\gamma^\mu\gamma^\nu - \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha$$

$$= 8\eta^{\alpha\beta}\eta^{\mu\nu} - 2\eta^{\beta\nu}\mathrm{Tr}\,\gamma^\mu\gamma^\alpha + \mathrm{Tr}\,\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\alpha$$

$$= 8\eta^{\alpha\beta}\eta^{\mu\nu} - 8\eta^{\beta\nu}\eta^{\mu\alpha} + 2\eta^{\mu\beta}\mathrm{Tr}\,\gamma^\nu\gamma^\alpha - \mathrm{Tr}\,\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha$$

$$= 8\eta^{\alpha\beta}\eta^{\mu\nu} - 8\eta^{\beta\nu}\eta^{\mu\alpha} + 8\eta^{\mu\beta}\eta^{\alpha\nu} - \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$$

$$\Rightarrow \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 4\left[\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\beta\nu}\eta^{\mu\alpha}\right]$$

Notation $\not{a} = a_\mu\gamma^\mu$

$\mathrm{Tr}\,\not{a} = 0$

$\mathrm{Tr}\,\not{a}\not{b} = 4a\cdot b$

$\mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d} = 4[a\cdot b\,c\cdot d + a\cdot d\,b\cdot c - a\cdot c\,b\cdot d]$

Quantizing Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\psi}} = 0 \Rightarrow (i\partial\!\!\!/ - m)\psi = 0$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \bar{\psi}i\gamma^0 = i\psi^\dagger$$

$$H = \int d^3x\,(\pi\dot{\psi} - \mathcal{L})$$

$$= \int d^3x\,(i\psi^\dagger\dot{\psi} - i\psi^\dagger\dot{\psi} - i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi)$$

$$= \int d^3x\,(-i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi)$$

Heisenberg field solves field eq.

$$(i\partial\!\!\!/ - m)\psi = 0$$

Try plane wave soln $\psi(x) = u(p)e^{-ip\cdot x}$

Note $(i\partial\!\!\!/ - m)\psi = 0$

$$\Rightarrow (i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi = 0$$

$$\Rightarrow (-\partial^2 + m^2)\psi = 0$$

So each component obeys KG eq. $\Rightarrow p^2 = m^2$.

To find spinor $u(p)$ must solve

$$(\not{p}-m)u(p)=0$$

Lets first work in rest frame $p^\mu=\hat{p}^\mu=(m,\vec{0})$

$$(m\gamma^0-m)u(\hat{p})=0$$

$$m\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}u(\hat{p})=0$$

$$\Rightarrow \quad u(\hat{p})=\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

convention

Under rotation ξ transforms like ordinary Pauli spinor. $\xi=\begin{pmatrix}1\\0\end{pmatrix}$, $\xi=\begin{pmatrix}0\\1\end{pmatrix}$ are usual spin up & spin down spinors. Lets get a more general solution by boosting along $\hat{z}$ axis. Recall ordinary boost corresponding to hyperbolic angle ϕ

$$\begin{pmatrix}E\\ p^3\end{pmatrix}=\begin{pmatrix}\cosh\phi & \sinh\phi\\ \sinh\phi & \cosh\phi\end{pmatrix}\begin{pmatrix}m\\ 0\end{pmatrix}=\begin{pmatrix}m\cosh\phi\\ m\sinh\phi\end{pmatrix}$$

Can use properties of hyperbolic function to show.

$$\sqrt{m}\cosh(\phi/2)=\frac{1}{2}\left(\sqrt{E+p^3}+\sqrt{E-p^3}\right)$$

$$\sqrt{m}\sinh(\phi/2)=\frac{1}{2}\left(\sqrt{E+p^3}-\sqrt{E-p^3}\right)$$

Now from our work on the Lorentz Group

$$u(\varphi) = D(\Lambda)\, u(\hat{p}) = \begin{pmatrix} e^{-(\phi\sigma^3/2)} & 0 \\ 0 & e^{(\phi\sigma^3/2)} \end{pmatrix} u(\hat{p})$$

$$= \exp\left(-\frac{\phi}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix}\right) u(\hat{p})$$

$$= \left[\sum_{n=0}^{\infty} \left(\frac{\phi}{2}\right)^{2n} \frac{1}{(2n)!} 1_{4\times 4} - \sum_{n=0}^{\infty} \left(\frac{\phi}{2}\right)^{2n+1} \frac{1}{(2n+1)!} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] u(\hat{p})$$

$$= \left[\cos\left(\frac{\phi}{2}\right) 1_{4\times 4} - \sinh\left(\frac{\phi}{2}\right) \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] u(\hat{p})$$

$$= \sqrt{m} \left(\begin{array}{c|c} \cosh\phi/2 - \sinh\phi/2\, \sigma^3 & 0 \\ \hline 0 & \cosh\left(\frac{\phi}{2}\right) + \sinh\left(\frac{\phi}{2}\right)\sigma^3 \end{array}\right) \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \sqrt{m} \begin{pmatrix} \left[\cosh\left(\frac{\phi}{2}\right) - \sigma^3 \sinh\left(\frac{\phi}{2}\right)\right] \xi \\ \left[\cosh\left(\frac{\phi}{2}\right) + \sigma^3 \sinh\left(\frac{\phi}{2}\right)\right] \xi \end{pmatrix}$$

$$= \begin{bmatrix} \left\{ \frac{1}{2}\left(\sqrt{E+p^3}+\sqrt{E-p^3}\right) - \frac{\sigma^3}{2}\left(\sqrt{E+p^3}-\sqrt{E-p^3}\right) \right\} \xi \\ \left\{ \frac{1}{2}\left(\sqrt{E+p^3}+\sqrt{E-p^3}\right) + \frac{\sigma^3}{2}\left(\sqrt{E+p^3}-\sqrt{E-p^3}\right) \right\} \xi \end{bmatrix}$$

There is a much more compact way to write the
Defini

$$\sigma^\mu = (1, \sigma^i) \quad , \quad \bar{\sigma}^\mu = (1, -\sigma^i)$$

$$p\cdot\sigma = E - \vec{\sigma}\cdot\vec{p} \quad , \quad p\cdot\bar{\sigma} = E + \vec{\sigma}\cdot\vec{p}$$

Now first row when squared

$$\left\{ \frac{1}{2}\left(\sqrt{E+p^3}+\sqrt{E-p^3}\right) - \frac{\sigma^3}{2}\left(\sqrt{E+p^3}-\sqrt{E-p^3}\right) \right\}^2$$

$$= \left\{ \sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right) \right\}^2$$

$$= (E+p^3)\left(\frac{1-\sigma^3}{2}\right) + (E-p^3)\left(\frac{1+\sigma^3}{2}\right)$$

$$= E - p^3\cdot\sigma^3 = p\cdot\sigma$$

Similarly square of second row is $p\cdot\bar{\sigma}$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \longleftarrow \text{true for general } p!$$