Physics 205

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Website: http://www.theory.caltech.edu/classes/ph205a

Outline for 1'st quarter

(1) Quantum Field Theory for Non Relativistic Systems

(2) Classical Field Theory: Symmetries + Conservation Laws

(3) Klein Gordon (Scalar) Field Theory

(4) Lorentz Invariance + Representations of the Lorentz Group

(5) Spinors, Dirac Matrices, Dirac Equation

(6) Quantization of Spin 1/2 fields, "antiparticles"

(7) Discrete Space-time symmetries, P, C, T

(8) Quantization of Electromagnetic field

Text: An Introduction to Quantum Field Theory
M.E. Peskin + D.V. Schroeder, Addison-Wesley 1995

Prereq: Classical Mechanics (field theory)
Quantum Mechanics
Representations of Rotation Group SU(2) algebra

Grade: P/F 100% problem sets

Units

In this course will set $\hbar = c = 1$

$c \sim l/t$, $\hbar \sim \text{energy} \times t$

$\hbar = c = 1$ $\quad l \sim t \sim 1/\text{energy} \sim 1/\text{mom} \sim 1/\text{mass}$

Can always put $\hbar, c$ factors in at very end.

Eg.

Suppose energy density of ... vacuum

$$\rho \simeq (10^{-3}\,\text{eV})^4 \simeq (10^{-9}\,\text{MeV})^4$$

in $\hbar = c = 1$ "units". But this is really an energy per unit volume

$$\rho = (10^{-9}\,\text{MeV})^4 / \underbrace{\hbar^3}_{(\text{MeV-sec})^3} \ \underbrace{c^3}_{(\text{cm/sec})^3}$$

$\hbar = 6.582 \times 10^{-22}$ MeV-sec
$c = 2.998 \times 10^{10}$ cm/sec

$$\rho = \frac{(10^{-9})^4}{(10^{-22})^3 (10^{10})^3} \frac{1}{(6.582 \times 2.998)^3}$$

$$\simeq 1.3 \times 10^{-4}\ \text{MeV/cm}^3$$

~ dark energy density in the universe

Many Particle Quantum Mechanics

Collection of N identical non relativistic particles interacting via potential

$$H = \sum_{K=1}^{N} T(x_K) + \frac{1}{2} \sum_{K \neq \ell} V(x_K, x_\ell)$$

$$T(x_K) = -\frac{1}{2m} \frac{d^2}{dx_K^2}$$

Pretend 1 spatial dimension to simplify notation trivial to then generalize to 3-dimensions.

$$i \frac{\partial}{\partial t} \Psi(x_1, \ldots x_N; t) = H \Psi(x_1 \ldots x_N; t)$$

Convenient to expand many-body wavefunction in a basis that is a product of single particle wave functions. Assume orthonormal basis

$$\psi_{q_K}(x_K)$$

q_K ← complete set of quantum numbers (eg momentum)

$$\Psi(x_1, \ldots x_N; t) = \sum_{q_1' \ldots q_N'} C(q_1', \ldots q_N'; t) \psi_{q_1'}(x_1) \ldots \psi_{q_N'}(x_N)$$

Insert in Schrodinger eq + multiply by $\psi^*_{q_1}(x_1) \ldots \psi^*_{q_N}(x_N)$ + integrate over $dx_1 \ldots dx_N$

$$i\frac{\partial}{\partial t} C(q_1 \dots q_N; t) = \sum_{k=1}^{N} \sum_{Q} \int dx_k\, \psi^*_{q_k}(x_k) T(x_k) \psi_Q(x_k)$$

$$\cdot C(q_1, \dots q_{k-1}, Q, q_{k+1} \dots q_N; t)$$

$$+\frac{1}{2} \sum_{k \neq l} \sum_{Q, Q'} \int dx_k \int dx_l\, \psi^\dagger_{q_k}(x_k)\, \psi^\dagger_{q_l}(x_l)\, V(x_k, x_l)$$

$$\psi_Q(x_k)\, \psi_{Q'}(x_l)\, C(q_1 \dots q_{k-1}, Q, q_{k+1} \dots q_{l-1}, Q', q_{l+1} \dots q_N; t)$$

Now particles identical

$$\psi(\dots x_i \dots x_j \dots) = \psi(\dots x_j \dots x_i \dots)$$

$$\Rightarrow C(\dots, q_i \dots q_j \dots) = C(\dots q_j \dots q_i \dots)$$

All that matters is # of times each quantum # occurs.

Out of set of quantum #'s $q_1 \dots q_l$ say
"1" occurs n_1 time
"2" occurs n_2 times

$$C(\underbrace{1 \dots 1}_{n_1}, \underbrace{2 \dots 2}_{n_2} \dots ; t)$$

$$\equiv \overline{C}(n_1, n_2 \dots n_\infty; t)$$

$$\sum_i n_i = N$$

Lets see what normalization constraint is

$$1 = \int dx_1 \ldots dx_N \, |\psi(x_1 \ldots x_N; t)|^2$$

$$= \sum_{q_1 \ldots q_N} |C(q_1 \ldots q_N; t)|^2$$

Now in sum over $q_1 \ldots q_N$ term corresponding to $(n_1 \ldots n_\infty)$ occurs

$$\frac{N!}{n_1! \, n_2! \ldots n_\infty!}$$

$\left\{ \begin{array}{l} \text{eg } N = 3 \quad (n_1 = 1, n_2 = 2) \\ (1,2,2), \ (2,1,2), \ (2,2,1) \\ \frac{3!}{2!\,1!} = 3 \end{array} \right.$

$$1 = \sum_{\substack{n_1 \ldots n_\infty \\ (\sum_i n_i = N)}} |\bar{C}(n_1 \ldots n_\infty; t)|^2 \frac{N!}{n_1! \, n_2! \ldots n_\infty!}$$

$(\sum_i n_i = N)$ ← usually omit

So usually introduce

$$f(n_1, \ldots n_\infty; t) = \sqrt{\frac{N!}{n_1! \ldots n_\infty!}} \, \bar{C}(n_1 \ldots n_\infty; t)$$

$$\sum_{n_1 \ldots n_\infty} |f(n_1 \ldots n_\infty)|^2 = 1$$

$$\psi(x_1, \ldots x_N; t) = \sum_{q_1 \ldots q_N} C(q_1, \ldots q_N; t) \, \psi_{q_1}(x_1) \ldots \psi_{q_N}(x_N)$$

$$= \sum_{q_1 \ldots q_N} \bar{C}(n_1, \ldots n_\infty; t) \, \psi_{q_1}(x_1) \ldots \psi_{q_N}(x_N)$$

$$= \sum_{n_1 \ldots n_\infty} f(n_1 \ldots n_\infty; t) \sqrt{\frac{n_1! \ldots n_\infty!}{N!}} \sum_{\substack{q_1 \ldots q_N \\ (n_1 \ldots n_\infty)}} \psi_{q_1}(x_1) \ldots \psi_{q_N}(x_N)$$

$$\Psi(x_1 \ldots x_N; t) = \sum_{n_1 \ldots n_\infty} f(n_1 \ldots n_\infty; t)\, \Phi_{n_1 \ldots n_\infty}(x_1 \ldots x_N)$$

$$\Phi_{n_1 \ldots n_\infty}(x_1 \ldots x_N) = \sqrt{\frac{n_1! \ldots n_\infty!}{N!}} \sum_{\substack{q_1 \ldots q_N \\ (n_1 \ldots n_\infty)}} \psi_{q_1}(x_1) \ldots \psi_{q_N}(x_N)$$

$\Phi_{n_1 \ldots n_\infty}(x_1 \ldots x_N)$ are o.n. many body wf's. They are o.n

$$\int dx_1 \ldots dx_N\, \Phi^*_{n_1' \ldots n_\infty'}(x_1, \ldots, x_N)\, \Phi_{n_1 \ldots n_\infty}(x_1, \ldots, x_N)$$

$$= \sqrt{\frac{n_1'! \ldots n_\infty'!}{N!}} \sqrt{\frac{n_1! \ldots n_\infty!}{N!}} \sum_{\substack{q_1' \ldots q_N' \\ (n_1' \ldots n_\infty')}} \sum_{\substack{q_1 \ldots q_N \\ (n_1 \ldots n_\infty)}} \int dx_1 \ldots dx_N$$

$$\psi^*_{q_1'}(x_1) \ldots \psi^*_{q_N'}(x_N)\, \psi_{q_1}(x_1) \ldots \psi_{q_N}(x_N)$$

$$= \frac{n_1! \ldots n_\infty!}{N!} \sum_{\substack{q_1 \ldots q_N \\ (n_1 \ldots n_\infty)}} 1\, \delta_{n_1 n_1'} \ldots \delta_{n_\infty n_\infty'}$$

$$= \delta_{n_1 n_1'} \ldots \delta_{n_\infty n_\infty'}$$

O.K with these tools lets go back to the Schrodinger eq. Focus on kinetic energy term

$$i \frac{\partial}{\partial t} C(q_1 \ldots q_N; t) = \sum_{k=1}^{N} \sum_{Q} \langle q_k | T | Q \rangle \cdot C(q_1 \ldots q_{k-1}, Q, q_{k+1} \ldots q_N; t)$$

+ potential term

Want to write it in terms of f's. First $\bar{C}(n_1 \dots n_\infty;t)$

LHS is pretty easy

$$i\frac{\partial}{\partial t} C(q_1 \dots q_N;t) = i\frac{\partial}{\partial t}\bar{C}(n_1 \dots n_\infty;t)$$

The kinetic term is a little trickier

$$KT = \sum_{k=1}^{N} \sum_{Q} \langle q_k | T | Q \rangle C(q_1 \dots q_{k-1}, Q, q_{k+1} \dots q_N;t)$$

$\langle q_k | T | Q\rangle$ same value for all q_k's that are the same. So replace in the $\sum_{k=1}^{N} \langle q_k | T | Q\rangle$ with $\sum_q n_q \langle q | T | Q\rangle$. Then

$$KT = \sum_q \sum_Q \langle q | T | Q\rangle n_q \bar{C}(n_1 \dots n_q - 1, \dots n_Q + 1, \dots n_\infty;t)$$

We will break up double sum into terms where $q=Q$ and term where $q \neq Q$. So lets put it together now

$$i\left[\frac{n_1! \dots n_\infty!}{N!}\right]^{1/2} \frac{\partial}{\partial t} f(n_1 \dots n_\infty;t)$$

$$= \sum_{i \neq j} \langle i | T | j\rangle n_i \left[\frac{n_1! \dots (n_i - 1)! \dots (n_j + 1)! \dots n_\infty!}{N!}\right]^{1/2}$$

$$\cdot f(n_1 \dots n_i - 1, \dots, n_j + 1 \dots n_\infty;t)$$

$$+ \sum_i \langle i | T | i\rangle n_i \left[\frac{n_1! \dots n_\infty!}{N!}\right]^{1/2} f(n_1 \dots n_\infty;t)$$

+ potential term

$$\Rightarrow \; i\frac{\partial}{\partial t} f(n_1 \ldots n_\infty; t) = \sum_i \langle i|T|i\rangle f(n_1 \ldots n_\infty; t)\, n_i$$
$$+ \sum_{i \neq j} \sqrt{n_i}\sqrt{n_j+1}\, \langle i|T|j\rangle f(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_\infty; t)$$
$$+ \text{potential term}$$

Occupation Number Basis

Here we introduce abstract notion of state vectors corresponding to wf's $\Phi_{n_1 n_2 \ldots n_\infty}$. Write states as $|n_1 \ldots n_\infty\rangle$ and the

$$|\Psi(t)\rangle = \sum_{n_1 n_2 \ldots n_\infty} f(n_1 \ldots n_\infty; t)\, |n_1 \ldots n_\infty\rangle$$

Note this could include states with different total # of particles N. Recall

$$N = \sum_{i=1}^{\infty} n_i$$

In systems where number of particles is conserved this is of no advantage. But for relativistic systems where, for example, pair creation can occur it is crucial

$$\langle n_1' \ldots n_\infty' | n_1 \ldots n_\infty \rangle = \delta_{n_1' n_1} \ldots \delta_{n_\infty' n_\infty}$$

We introduce an operator b_k that removes a particle with quantum number "k".

$$b_k |n_1 \ldots n_k \ldots n_\infty\rangle = \sqrt{n_k}\, |n_1 \ldots, n_k - 1, \ldots n_\infty\rangle$$

$$b_k |n_1 \ldots n_{k-1}, 0, n_{k+1} \ldots n_\infty\rangle = 0$$

state $|0, \ldots, 0\rangle$ is the no particle state, i.e. the vacuum. Sometimes abbreviate by just $|0\rangle$. It is normalized to unity $\langle 0|0\rangle = 1$.

To save writing, suppress "other quantum numbers"

$$b_k |n_k\rangle = \sqrt{n_k}\,|n_k - 1\rangle$$

$$\langle n_k' | b_k | n_k\rangle = \sqrt{n_k}\,\langle n_k' | n_k - 1\rangle$$

$$= \sqrt{n_k}\,\delta_{n_k', n_k - 1}$$

$$= \sqrt{n_k}\,\delta_{n_k' + 1, n_k}$$

$$= \sqrt{n_k' + 1}\,\delta_{n_k' + 1, n_k}$$

But $\langle n_k' | b_k | n_k\rangle = \langle b_k^\dagger n_k' | n_k\rangle$

$$\Rightarrow b_k^\dagger |n_k'\rangle = \sqrt{n_k' + 1}\,|n_k' + 1\rangle$$

So $b_k^\dagger$ adds a particle with quantum # k. It's a creation operator

$$b_k b_k^\dagger |n_k\rangle = \sqrt{n_k + 1}\, b_k |n_k + 1\rangle = (n_k + 1)\,|n_k\rangle$$

$$b_k^\dagger b_k |n_k\rangle = \sqrt{n_k}\, b_k^\dagger |n_k - 1\rangle = n_k\,|n_k\rangle$$

So $b_k^\dagger b_k$ counts # of particles with quantum # k

$$[b_k, b_k^\dagger] = b_k b_k^\dagger - b_k^\dagger b_k = 1$$

$$[b_k, b_{k'}^\dagger] = \delta_{kk'}$$

$$[b_k, b_{k'}] = [b_k^\dagger, b_{k'}^\dagger] = 0$$