Perturbation Theory Fn Cross Sections

We showed last lecture that in $\lambda\phi^4$ theory

$$\langle \Omega | T \{ \underbrace{\phi(x)}_{\text{Full Heisenberg Field}} \phi(y) \} | \underbrace{\Omega}_{\text{true groundstate}} \rangle$$

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \exp[-i \int_{-T}^{T} d\tau\, H_I(\tau)] \} | 0 \rangle}{\langle 0 | T \{ \exp[-i \int_{-T}^{T} d\tau\, H_I(\tau)] \} | 0 \rangle}$$

Easy to convince yourself it generalizes to more fields

$$\langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle$$

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x_1) \cdots \phi_I(x_n) \exp[-i \int_{-T}^{T} d\tau\, H_I(\tau)] \} | 0 \rangle}{\langle 0 | T \{ \exp[-i \int_{-T}^{T} d\tau\, H_I(\tau)] \} | 0 \rangle}$$

Where in above formulas

$$H_I(\tau) = \frac{\lambda}{4!} \int d^3x\, \phi_I^4(\vec{x}, \tau)$$

Fields $\phi_I(x)$ are interaction picture fields. As we noted last quarter they are the free fields you know well. They have the usual mode expansion

$$\phi_I(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \left(a(k) e^{-ik\cdot x} + a(k)^\dagger e^{ik\cdot x} \right) \qquad k\cdot x = k^0 x^0 - \vec{k}\cdot\vec{x}$$

$$k^0 = E_k = \sqrt{\vec{k}^2 + m^2}.$$

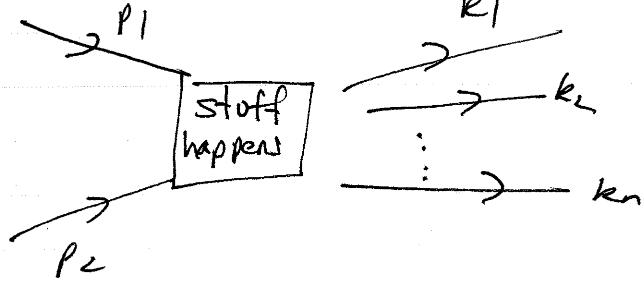
State $|0\rangle$ is annihilated by $a(k)$'s; $a(k)|0\rangle = 0$ and is normalized to unity $\langle 0|0\rangle$. We will relate cross sections + particle masses to these expectation values

Aside on Cross Section

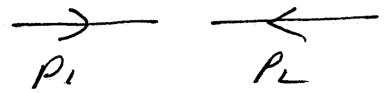
Transition Matrix Element

$S_{fi} = \langle f | i \rangle$

$|i\rangle$ is Heisenberg state that in the infinite past has well separated particles with momentum p_1, p_2 and $\langle f|$ is state in "future infinity" has well separated particles with momenta $k_1 \ldots k_n$



Want to derive formula for differential cross section. Go to head on collision frame



$$d\sigma_{fi} = \frac{dw_{fi}}{N\, J_{inc}}$$

N = # of target particles per unit volume $= \dfrac{1}{V}$ ← regulating volume

J = incident flux $\qquad J = \dfrac{|\vec{v}_1 - \vec{v}_2|}{V}$

dw_{fi} = transition rate per unit Volume

$$d\sigma \sim V^2 \frac{1}{VT} \underset{\hbar = c = 1}{\sim} L^2$$

According to principles of QM

per unit volume

$$dW_{fi} = \frac{\left(|\langle f|i\rangle|^2/VT\right)}{2E_{p_1}V\, 2E_{p_2}V\, 2E_{k_1}V \cdots 2E_{k_n}V} \quad \frac{Vd^3k_1}{(2\pi)^3} \cdots \frac{Vd^3k_n}{(2\pi)^3}$$

↳ # of states in range d^3k_i; $\vec{n} = \frac{L\vec{k}}{2\pi}$

There we have used normalization of state

$$\langle \vec{p}|\vec{p}\rangle = 2E_p (2\pi)^3 \delta^3(0) \rightarrow 2E_p V$$

finite volume

$$(2\pi)^3 \delta^3(\vec{p}) = \int d^3x\, e^{-i\vec{p}\cdot x} \xrightarrow[\vec{p}\to 0]{} (2\pi)^3 \delta^3(0) = \int d^3x\, 1 = V$$

So putting it all together

$$dW_{fi} = \frac{\left(|\langle i|f\rangle|^2/VT\right)}{2E_{p_1}\, 2E_{p_2}} \quad \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2E_{k_n}}$$

Because interactions conserve momenta

$$\langle f|i\rangle = (2\pi)^4 \delta^4(p_1 + p_2 - k_1 \ldots - k_n)\, M_{fi}$$

$$|\langle f|i\rangle|^2 = (2\pi)^4 \delta^4(p_1 + p_2 - k_1 \ldots - k_n)(2\pi)^4 \delta^4(0)\, |M_{fi}|^2$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - k_1 \ldots - k_n)\, VT\, |M_{fi}|^2$$

$$d\sigma_{fi} = \frac{|M_{fi}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 \ldots - k_n)}{2E_{p_1} 2E_{p_2} |\vec{v}_1 - \vec{v}_2|} \quad \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2E_{k_n}}$$

But $\vec{v} = p/E$ $\qquad \begin{pmatrix} E = m\gamma \\ p = mv\gamma \end{pmatrix} \Rightarrow v = p/E$

$$2E_{p_1} 2E_{p_2} |\vec{v}_1 - \vec{v}_2| = 4E_{p_1}E_{p_2}\overbrace{(|\vec{v}_1| + |\vec{v}_2|)}^{\text{headon}}$$

$$= 4\left(E_{p_2}|\vec{p}_1| + E_{p_1}|\vec{p}_2|\right)$$

But $(p_1 \cdot p_2)^2 - m_1^2 m_2^2$

$$= \left(E_{p_1}E_{p_2} + |\vec{p}_1||\vec{p}_2|\right)^2 - \left(E_{p_1}^2 - \vec{p}_1^{\,2}\right)\left(E_{p_2}^2 - \vec{p}_2^{\,2}\right)$$

headon

$$= \cancel{E_1^2 E_2^2} + 2|\vec{p}_1||\vec{p}_2|E_1E_2 + \cancel{\vec{p}_1^{\,2}\vec{p}_2^{\,2}} - \cancel{E_{p_1}^2 E_{p_2}^2}$$
$$+ \vec{p}_1^{\,2}E_{p_2}^2 + p_2^2 E_{p_1}^2 \cancel{- p_1^2 p_2^2}$$

$$= \left(E_{p_2}|\vec{p}_1| + E_{p_1}|\vec{p}_2|\right)^2$$

So
$$2E_{p_1}2E_{p_2}|\vec{v}_1 - \vec{v}_2| = 4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}$$
In any frame

$$d\sigma_{fi} = \frac{|M_{fi}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 \ldots - k_n)}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2E_{k_n}}$$

$${}_{\text{out}}\langle k_1 \cdots k_n | p_1 p_2\rangle_{\text{in}} = M_{fi}(2\pi)^4 \delta^4(p_1 + p_2 - k_1 - \ldots - k_n)$$

For $2 \to 2$ scattering most of phase space integrals done by delta functions without knowing explicit form of M_{fi}. Consider phase space integral for $p_1 + p_2 \to k_1 + k_2$

$$PS = \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \int \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2)$$

We will do this in general not restricting initial + final state particles to have same mass. Initial particle with masses $m_{1,2}$ + final with mass $\mu_{1,2}$.

$$(p_1+p_2)^2 = s \quad \text{In cm frame} \quad p_1+p_2 = (\sqrt{s}, \vec{0})$$

$$PS = \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \int \frac{d^4k_2}{(2\pi)^4} \delta(k_2^2-\mu_2^2)(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$$

$$= \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \delta\left((p_1+p_2-k_1)^2-\mu_2^2\right)$$

$$d^3k_1 = d\Omega_1 |\vec{k}_1|^2 d|\vec{k}_1|$$

$$dE_{k_1} = d\sqrt{|\vec{k}_1|^2+\mu_1^2} = \frac{|\vec{k}_1|\, d|\vec{k}_1|}{E_{k_1}}$$

$$\Rightarrow E_{k_1} dE_{k_1} = |\vec{k}_1|\, d|\vec{k}_1|$$

cm frame

$$PS = \int \frac{d\Omega_1}{(2\pi)^2} \frac{|\vec{k}_1|\, dE_{k_1}}{2} \delta\left(s+\mu_1^2-\mu_2^2-2\sqrt{s}E_{k_1}\right)$$

$$= \frac{|\vec{k}_1|}{(2\pi)^2 4\sqrt{s}} \int d\Omega_1 \qquad E_{k_1} = \frac{s+\mu_1^2-\mu_2^2}{2\sqrt{s}}$$

Now need

$$4E_{p_1}E_{p_2}|V_1 - V_2| = 4(|p_1|E_{p_2} + |p_2|E_{p_1})$$

$$= \underset{\uparrow \atop \text{cms}}{4|p_1|}(E_{p_1} + E_{p_2}) = 4|\vec{p}_1|\sqrt{s}$$

$$d\sigma = \frac{|M_{fi}|^2}{64\pi^2\sqrt{s}}\left(\frac{|\vec{k}_1|}{|\vec{p}_1|}\right)$$

Energy + momentum conservation

$$E_{k_1} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \qquad E_{k_2} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$

$$E_{p_1} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \qquad E_{p_2} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$