Can write the Schroedinger eq in terms of creation & annihilation operators acting on this abstract Hilbert space.

Introduce

$$\hat{T} = \sum_{ij} b_i^\dagger \langle i|T|j\rangle b_j$$

$$\hat{V} = \frac{1}{2}\sum_{ijkl} b_i^\dagger b_j^\dagger \langle ij|V|kl\rangle b_k b_l$$

$$\langle i|T|j\rangle = \int dx\, \varphi_i^*(x)\, T(x)\, \varphi_j(x)$$

$$\langle ij|V|kl\rangle = \int dx\, dy\, \varphi_i^*(x)\, \varphi_j^*(y)\, V(|x-y|)\, \varphi_k(x)\, \varphi_l(y)$$

Claim Schroedinger eq is

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = (\hat{T}+\hat{V})\,|\Psi(t)\rangle$$

Check explicitly for kinetic term

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \sum_{n_1'\cdots n_\infty'} i\frac{\partial}{\partial t} f(n_1'\cdots n_\infty';t)\,|n_1'\cdots n_\infty'\rangle$$

$$= \sum_{n_1'\cdots n_\infty'}\sum_{ij} b_i^\dagger \langle i|T|j\rangle b_j\, |n_1'\cdots n_\infty'\rangle f(n_1'\cdots n_\infty';t)$$

+ potential term

Multiply by $\langle n_1\cdots n_\infty|$

$$i\frac{\partial}{\partial t} f(n_1\cdots n_\infty;t) = \sum_{n_1'\cdots n_\infty'}\sum_{ij} \langle n_1\cdots n_\infty|\, b_i^\dagger b_j\,|n_1'\cdots n_\infty'\rangle \langle i|T|j\rangle$$

$$f(n_1'\cdots n_\infty';t) \quad + \text{potential term}$$

$$= \sum_{i} n_i \langle i|T|i\rangle f(n_1\cdots n_\infty;t) + \sum_{i\neq j} \sqrt{n_i}\,\sqrt{n_j+1}$$

$\uparrow_{i=j}$

$$\cdot f(n_1 \cdots n_i - 1, \cdots n_j + 1 \cdots n_\infty;t)$$

+ potential term

Quantum Fields

Introduce quantum fields

$$\phi(x) = \sum_k \psi_k(x) b_k$$

single particle wf's

quantum field

$$\phi^\dagger(x) = \sum_k \psi_k^*(x) b_k^\dagger$$

Single particle wf's are a complete set

$$\psi(y) = \sum_k a_k \psi_k(y)$$

$$a_k = \int dx\, \psi_k^*(x) \psi(x)$$

$$\psi(y) = \sum_k \int dx\, \psi_k^*(x) \psi_k(y)\, \psi(x)$$

$$= \int dx \left(\sum_k \psi_k^*(x) \psi_k(y) \right) \psi(x)$$

So $\sum_k \psi_k^*(x)\, \psi_k(y) = \delta(x-y)$

Commutator of two quantum fields

$$[\phi(x), \phi^\dagger(y)] = \sum_{k,l} \psi_k(x) \psi_l^*(y) [b_k, b_l^\dagger]$$

$$= \sum_{k,l} \psi_k(x) \psi_l^*(y)\, \delta_{kl}$$

$$= \sum_k \psi_k^*(y) \psi_k(x) = \delta(x-y)$$

But $[\phi(x), \phi(y)] = [\phi^\dagger(x), \phi^\dagger(y)] = 0$

$\phi^\dagger(x)$ creates particle at pt x from vacuum

$\langle y|\phi^\dagger(x)|0\rangle = \sum_k \psi_k^*(x)\langle y|k\rangle = \sum_k \psi_k^*(x)\psi_k(y) = \delta(x-y)$

In terms of quantum fields

$$\hat{H} = \int dx\, \phi^\dagger(x)\, T(x)\, \phi(x)$$

$$+\frac{1}{2}\int dx\,dy \quad \phi^\dagger(x)\phi^\dagger(y)\, V(|x-y|)\, \phi(y)\phi(x)$$

For example,

$$\int dx \qquad \phi^\dagger(x)\, T(x)\, \phi(x)$$

$$= \sum_k \sum_\ell \int dx \;\, b_k^\dagger\, \psi_k^*(x)\, T(x)\, \psi_\ell(x)\, b_\ell$$

$$= \sum_{k,\ell} \; b_k^\dagger \langle k|T|\ell\rangle b_\ell$$

Other operators done similarly. Number density of particles

$$n(x) = \sum_k \delta(X - x_k)$$

$$\int_V n(x)\, dX = \#\text{ of particles in vol } V$$

Quantum mechanical operator.

$$\hat{n}(x) = \int dy\;\, \phi^\dagger(y)\, \delta(x-y)\, \phi(y)$$

$$= \phi^\dagger(x)\,\phi(x)$$

Also $\quad \hat{n}(x) = \sum_{i,j} b_i^\dagger \langle i|n(x)|j\rangle b_j$

where $\quad \langle i|n(x)|j\rangle = \int dy \;\, \psi_i^*(y)\, \delta(x-y)\, \psi_j(y)$

$$= \psi_i^*(x)\, \psi_j(x)$$

total number of particles

$$\hat{N} = \int dx\, \phi^\dagger(x)\phi(x)$$

For our Hamiltonian the total number of particles is conserved

$$[\hat{N}, H] = 0$$

Free field Theory

Suppose particles have no internal quantum number (spin zero) and are identical + non interacting ($V=0$). Then Non relativist Hamiltonian

$$H = \sum_k -\frac{\vec{\nabla}_k^2}{2m}$$ ← Back to 3-dimension

$$H = -\int d^3x\, \phi^\dagger(x)\frac{\vec{\nabla}^2}{2m}\phi(\vec{x})$$

Imagine putting the system in a box with sides of length L. Suppose we impose periodic bc's. For large box shouldn't matter. Choose 1-particle quantum numbers that diagonalize the Hamiltonian. Because of box discrete.

$$\psi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{L^3}} e^{i\vec{k}\cdot\vec{x}} \qquad \vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3) = \frac{2\pi}{L}\vec{n}$$ ← integers

$$\phi(\vec{x}) = \sum_{\vec{k}} b_{\vec{k}}\, \psi_{\vec{k}}(\vec{x})$$

Want to take infinite volume limit $L \to \infty$

$$\sum_{\vec{n}} \rightarrow \int d^3n = \left(\frac{L}{2\pi}\right)^3 \int d^3k$$

Let $L^{3/2} b_{\vec{k}} \rightarrow b(\vec{k})$.
So in large volume limit

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, b(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

What are the commutation relations for $b(\vec{k})$

$$[b(\vec{k}), b(\vec{k}')^{+}] = L^3 \delta_{\vec{n},\vec{n}'}$$

$$\sum_{\vec{n}} \delta_{\vec{n}\vec{n}'} = 1$$

$$\left(\frac{L}{2\pi}\right)^3 \int d^3k\, \delta_{\vec{n},\vec{n}'} = 1$$

$$\left(\frac{L}{2\pi}\right)^3 \delta_{\vec{n},\vec{n}'} \longrightarrow \delta^3(\vec{k}-\vec{k}')$$

$$[b(\vec{k}), b^{+}(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$[b(\vec{k}), b(\vec{k}')] = [b^{+}(\vec{k}), b^{+}(\vec{k}')] = 0$$

Introduce the no particle state annihilated by $b(\vec{k})$

$$b(\vec{k})|0\rangle = 0$$

and normalized to unit

$$\langle 0|0\rangle = 1$$

The one particle state

$$|\vec{k}\rangle = b^{+}(\vec{k})|0\rangle$$

$$\langle \vec{k}'|\vec{k}\rangle = \langle b^{+}(\vec{k}')0|b^{+}(\vec{k})0\rangle$$

$$= \langle 0|b(\vec{k}')b^{+}(\vec{k})|0\rangle$$

$$= \langle 0|[b(\vec{k}'), b^{+}(\vec{k})]|0\rangle$$

$$= (2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

Similarly

$$b(\vec{k}')|\vec{k}\rangle = c\,\delta^3(\vec{k}-\vec{k}')|0\rangle$$

for some constant c

$$\langle 0|b(\vec{k}')|\vec{k}\rangle = c\,\delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow \langle \vec{k}'|\vec{k}\rangle = c\,\delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow \quad c = (2\pi)^3$$

$$H = -\int d^3x\, \phi^{+}(\vec{x})\frac{\vec{\nabla}^2}{2m}\phi(\vec{x})$$

$$= -\int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\left(\frac{-\vec{k}^2}{2m}\right)e^{-i\vec{k}'\cdot\vec{x}}e^{i\vec{k}\cdot\vec{x}}\,b^{+}(\vec{k}')b(\vec{k})$$

$\int d^3x$ gives $(2\pi)^3\delta^3(\vec{k}-\vec{k}')$ + use Dirac delta function to do k' integration

$$\boxed{H = \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m} b^\dagger(k)\, b(\vec{k})}$$

Now

$$H\,|\vec{k}\rangle = \int \frac{d^3\vec{k}'}{(2\pi)^3} \left(\frac{\vec{k}'^2}{2m}\right) b^\dagger(\vec{k}')\, b(\vec{k}')\, |\vec{k}\rangle$$

$$= \int \frac{d^3\vec{k}'}{(2\pi)^3} \left(\frac{\vec{k}'^2}{2m}\right) b^\dagger(\vec{k}') \underbrace{\left[b(\vec{k}'), b^\dagger(\vec{k})\right]}_{(2\pi)^3\delta^3(\vec{k}-\vec{k}')} |0\rangle$$

$$= \frac{\vec{k}^2}{2m} b^\dagger(\vec{k})\,|0\rangle$$

$$= \frac{\vec{k}^2}{2m} |\vec{k}\rangle$$

So $|\vec{k}\rangle$ is an eigenstate of H with energy eigenvalue $E_{\vec{k}} = \vec{k}^2/2m$. Get rest of eigenstates by acting more times with $b^\dagger$'s.

So far everything was in the Schrodinger picture. Lets go over to the Heisenberg picture. To do that use (for any operator O)

$$O_H(t) = e^{iHt}\, O\, e^{-iHt}$$

$\uparrow$ Schroedinger operator

$$i\frac{\partial}{\partial t} O_H(t) = [O_H, H] \qquad O_H(0) = O$$

Note if $[O, H] = X \Rightarrow [O_H, H] = X_H$

For $b(\vec{k})$

$$[b(\vec{k}), H] = \left[b(\vec{k}), \int \frac{d^3k'}{(2\pi)^3} \frac{\vec{k}'^2}{2m} b^+(\vec{k}')\, b(\vec{k}')\right]$$

$$= \frac{\vec{k}^2}{2m} b(\vec{k})$$

$$[b_H(\vec{k},t), H] = \frac{\vec{k}^2}{2m} b_H(\vec{k},t)$$

$$\Rightarrow \quad \frac{i\partial}{\partial t} b_H(\vec{k},t) = \frac{\vec{k}^2}{2m} b_H(\vec{k},t)$$

$$b_H(\vec{k},t) = e^{-iE_k t} b(\vec{k}) \quad , \quad E_k = \frac{\vec{k}^2}{2m}$$

So Heisenberg field is

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} b_H(\vec{k},t)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x} - E_k t)} b(\vec{k}) \qquad E_k = \frac{\vec{k}^2}{2m}$$

$$[\phi_H(\vec{x},t), \phi_H^+(\vec{y},t)] = \delta^3(\vec{x}-\vec{y})$$