Continuum Field Theory in Classical & Quantum Physics

Suppose we have a canonical coordinate centered in space \( \phi(x) \). Discretize space with lattice spacing \( \alpha \). Then continuum field theory becomes a more familiar field:

\[
L = \int d^3 x \; \mathcal{L} (\phi(x), \phi'(x))
\]

\[
\mathcal{L} = \sum_i \alpha^3 \; \mathcal{L} (\phi(x_i), \phi'(x_i))
\]

\[
\alpha^3 \frac{\partial \mathcal{L}}{\partial \phi'(x)} = \rho(x)
\]

Define canonical momentum density:

\[
\pi(x_i) = \rho(x_i) / \alpha^3
\]

\[
H = \sum_i \pi(x_i) \phi'(x_i) - L
\]

\[
\lim_{\alpha \to 0} \int d^3 x \; \left[ \pi(x) \phi'(x) - \mathcal{L} \right]
\]

\[
\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}
\]

\[
\text{To quantize, use } \left[ \rho(x_i), \phi(x_j) \right] = -\alpha \delta_{ij}
\]
\[ L \pi(x_i), \phi(x_j) = -i \alpha^3 \delta_{ij} \rightarrow -i \delta^3(x_i - x_j) \]

So in continuum field theory we have commutation relations:

\[ [\pi(\vec{x}), \phi(\vec{y})] = -i \delta^3(\vec{x} - \vec{y}) \]

\[ [\pi(\vec{x}), \pi(\vec{y})] = [\phi(\vec{x}), \phi(\vec{y})] = 0 \]

What is the Lagrangian density for the Non-Relativistic field theory?

\[ L = \int d^3x \left\{ i \phi^*(\vec{x}) \dot{\phi}(\vec{x}) + \phi^*(\vec{x}) \frac{\nabla^2 \phi(\vec{x})}{2m} \right\} \]

\[ = \int d^3x \left( -i \phi^*(\vec{x}) \dot{\phi}(\vec{x}) - L \right) \]

This is called sigma

\[ \Pi(\vec{x}) = \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = i \phi^*(\vec{x}) \]

\[ H = \int d^3x \left\{ \Pi(\vec{x}) \phi(\vec{x}) - L \right\} \]

\[ = -\int d^3x \phi^*(\vec{x}) \frac{\nabla^2 \phi(\vec{x})}{2m} \]

Commudal relashon between \( \Pi \) and \( \phi \) gives

\[ [\phi^*(\vec{x}, t), \phi(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) \]
**CLASSICAL Relativistic Conventions**

**Notation**

\[ \hbar = c = 1 \]

[\text{length}] \sim [\text{time}] \sim [\text{energy}]^{-1} \sim [\text{momentum}]^{-1}

**Space-time metric**

\[ \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

**Greek indices go over \( \{0,1,2,3\} \), roman over \( 1,2,3 \)**

\[ \eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu = \text{d}t^2 - \text{d}x^0 \text{d}x^0 = \text{d}t^2 - \text{d}x^1 \text{d}x^1 - \text{d}x^2 \text{d}x^2 - \text{d}x^3 \text{d}x^3 \]

\[ x^\mu = (x^0, \vec{x}) \]

**Inclines raised, lowered with \( \eta \)**

\[ x^1 \cdot x_2 = \eta_{\mu\nu} x^\mu x_\nu = x^0 x^0 - \vec{x}_1 \cdot \vec{x}_2 \]

**4-vector \( x^\mu \)**

\[ x^\mu = x^\mu_1 x^\mu_2 = x^\mu_1 x^\mu_2 \]

\[ x^\mu = (x^0, -\vec{x}) \]

**A particle of mass \( m \) has four momentum**

\[ \rho^\mu = (E, \vec{p}) \]

\[ \rho^2 = \eta_{\mu\nu} \rho^\mu \rho^\nu = E^2 - \vec{p}^2 = m^2 \]

\[ \partial^\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \vec{\nabla}) \]
Field Equation For Classical Field Theory

The action is defined as $S \equiv \int L \, dt$, where $L$ is the local field theory Lagrangian. The Lagrangian is defined over space $S$ and time $t$, and depends on fields $\xi^{\alpha}$ and their derivatives at each point in space.

$$S = \int dt \, L = \int dt \, d^3x \, L(\xi^\alpha, \partial_t \xi^\alpha)$$

$\xi^\alpha = \xi^\alpha(x)$. Principal of least action

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \delta S}{\partial \xi^\alpha} + \frac{\partial \delta S}{\partial (\partial_t \xi^\alpha)} \right\}$$

$$\delta \left( \partial_t \xi^\alpha \right) = 0$$

Repeated and all summed.

$$\delta \left( \partial_t \xi^\alpha \right) = \partial_t (\delta \xi^\alpha)$$

and

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \delta S}{\partial \xi^\alpha} + \partial_t \left[ \frac{\partial \delta S}{\partial \left( \partial_t \xi^\alpha \right)} \right] - \left[ \partial_t \left( \partial_t \xi^\alpha \right) \right] \delta \xi^\alpha \right\}$$
Second term added wherever can vanish anywhere on boundary of some $\Sigma$. According to usual principal values $L$ term + $t$. Also assume vanishes if $\phi = 0$, so shift second term + $cun \phi$ is arbitrary.

\[
\frac{\partial L}{\partial \phi^a} - \partial_t \left( \frac{\partial L}{\partial (\partial_t \phi^a)} \right) = 0
\]

\[
\frac{\partial L}{\partial \phi^a} - \partial_t \left( \frac{\partial L}{\partial (\partial_t \phi^a)} \right) - \partial_j \left( \frac{\partial L}{\partial (\partial_j \phi^a)} \right) = 0
\]

Called Euler-Lagrange Equation. Even though usual relativistic notation for derivative holds for non-relativistic systems.

Hamiltonian

Recall canonical momentum density

\[
\Pi^a = \frac{\partial L}{\partial (\partial_t \phi^a)}
\]

\[
H = \int d^3x \left\{ \Pi^a \phi^a - L \right\} = \int d^3x \mathcal{H}
\]

\[
\mathcal{H} = \Pi^a \dot{\phi}^a - L
\]
Example 1: Klein Gordon Field Theory

Consider a real scalar field $\phi(x)$. Scalar refers to how it transforms under Lorentz transformations. If Lagrangian density is a scalar, then you get a Lorentz invariant theory. We'll say more about Lorentz transformations later.

$$L = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right)$$

$$= \frac{1}{2} \left[ \phi^2 - (\nabla \phi)^2 - m^2 \phi^2 \right] + \text{time derivative}$$

$$\frac{\partial L}{\partial \phi} = \phi$$

$$\frac{\partial L}{\partial (\partial \phi)} = 0$$

$$-m^2 \phi - \partial_\mu \partial^\mu \phi = 0$$

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi = 0$$

The canonical momentum density

$$\Pi(x) = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}$$

The Hamiltonian density is

$$H = \Pi \dot{\phi} - L = \frac{1}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} \left( \phi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right)$$

$$H = \int d^3x \left( \frac{1}{2} \left( \phi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right) \right) \quad \text{positive}$$