A Fact From QM

Heisenberg Operators obey the classical equations of motion

Eg 1-dim, 1-particle system in QM

$$H = \frac{p^2}{2m} + V(x)$$

$$\mathcal{L} = \frac{\dot{x}^2}{2m} - V(x)$$

Eqs of motion

$$\dot{x} = \frac{p}{m} = \{x, H\}_{PB}$$

$$\{A, B\}_{PB} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

$$\dot{p} = -\frac{dV}{dx} = \{p, H\}_{PB}$$

Quantum mechanics these become

$$i\dot{X}_H = [X_H, H]$$

$$i\dot{p}_H = [p_H, H]$$

$$[X_H, p_H] = i$$

$$i\dot{X}_H = \left[X_H, \frac{p_H^2}{2m} + V(x_H)\right] = \frac{2[X_H, p_H]\,p_H}{2m} = \frac{2i}{2m}p_H$$

$$\dot{x}_H = p_H/m$$

$$i\dot{p}_H = \left[p_H, \frac{p_H^2}{2m} + V(x_H)\right]$$

Expand $V(x_H)$ in a power series about 0

$$V(x_H) = \sum_{n=0}^{\infty} \frac{V_n x_H^n}{n!}$$

$$[p_H, x_H^n] = n\, x_H^{n-1} [p_H, x_H] = -i n\, x_H^{n-1}$$

so $$[p_H, V(x_H)] = -i \frac{dV(x_H)}{dx_H}$$

$$\dot{x}_H = \frac{p_H}{m} \qquad \Rightarrow \quad \ddot{x}_H + \frac{dV(x_H)}{dx_H} = 0$$

$$\dot{p}_H = -\frac{dV(x_H)}{dx}$$

Continuum Field Theory in Classical + Quantum Physics

Suppose we have a canonical coordinate at each pt of space $\phi(\vec{x})$. Discretize space with a lattice of spacing "a". Then continuum field theory looks more familiar:

$$L = \int d^3x \; \mathcal{L}(\phi(\vec{x}), \dot{\phi}(\vec{x}))$$

continuum

for simplicity no gradients

$$L = \sum_i a^3 \mathcal{L}(\phi(x_i), \dot{\phi}(x_i))$$

$$\phi(x_i) \equiv q_i \qquad \text{canonical coordinate}$$

$$p_i = \frac{dL}{d\dot{q}_i} = a^3 \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x}_i)} \qquad p_i \equiv p(x_i)$$

$$H = \sum_i p(\vec{x}_i)\, \dot{\phi}(\vec{x}_i) - L$$

Define canonical momentum density

$\pi(x_i) = p(x_i)/a^3$

$H = \sum_i a^3 \pi(x_i)\, \dot\phi(x_i) - L$

$\rightarrow \int d^3x \left[\pi(x)\dot\phi(x) - \mathcal{L}\left(\phi(\vec{x}), \dot\phi(\vec{x})\right)\right]$

$\pi(\vec{x}) \rightarrow \partial\mathcal{L}/\partial\dot\phi$

Recall $[p_i, q_j] = -i\delta_{ij}$

$\Rightarrow [\pi(x_i), \phi(x_j)] = -i a^{-3}\delta_{ij} = -i\delta^3(\vec{x}_i - \vec{x}_j)$

Canonical commutation relation in field theory

$[\pi(\vec{x}), \phi(\vec{y})] = -i\delta^3(\vec{x}-\vec{y})$

$[\pi(x), \pi(y)] = [\phi(\vec{x}), \phi(\vec{y})] = 0$

What is Lagrange density for NR free field th

$L = \int d^3x \left\{ i\phi^\dagger(\vec{x})\dot\phi(\vec{x}) + \phi^\dagger(\vec{x})\frac{\vec{\nabla}^2}{2m}\phi(\vec{x}) \right\}$

$\pi(\vec{x}) = \dfrac{\partial\mathcal{L}}{\partial\dot\phi(\vec{x})} = i\phi^\dagger(\vec{x})$

$H = \int d^3x\ \pi(x)\dot\phi(\vec{x}) - \mathcal{L} = -\int d^3x\, \phi^\dagger(\vec{x})\frac{\vec{\nabla}^2}{2m}\phi(\vec{x})$

$[i\phi^\dagger(\vec{x}), \phi(\vec{y})] = -i\delta^3(\vec{x}-\vec{y})$

$\Rightarrow [\phi^\dagger(\vec{x}), \phi(\vec{y})] = -\delta^3(\vec{x}-\vec{y})$

Relativistic Notation

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Greek indices go over {0, 1, 2, 3} + roman over {1, 2, 3}. Repeated indices are summed unless explicitly saying otherwise

$$\eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^a dx^a = dt^2 - {dx^1}^2 - {dx^2}^2 - {dx^3}^2$$

$$x^\mu = (x^0, \bar{x})$$

Indices raised + lowered with η

$$x_1 \cdot x_2 = \eta_{\mu\nu} x_1^\mu x_2^\nu = x_1^0 x_2^0 - \vec{x}_1 \cdot \vec{x}_2$$

↖ 4-vector dot product

$$x_1 \cdot x_2 = x_{1\mu} x^{2\mu} = x_1^\mu x_{2\mu}$$

$$x_\mu = (x^0, -\bar{x})$$

A particle of mass m has four momentum

$$p^\mu = (E, \bar{p})$$

$$p^2 = \eta_{\mu\nu} p^\mu p^\nu = E^2 - \bar{p}^2 = m^2$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \bar{\nabla})$$

Field Equations For Classical Field Theory

The action is the integral of Lagrangian over time. In local field theory Lagrangian is integral over space of Lagrange density

$$S = \int dt\, L = \int dt\, d^3x \;\; \mathcal{L}(\phi^a, \partial_\mu \phi^a)$$

type of field

↳ depends on field time derivative + field space derivative.

$\phi^a \equiv \phi^a(x)$. Principal of least action

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_j \phi^a)} \delta(\partial_j \phi^a) \right.$$

$$\left. + \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta(\partial_t \phi^a) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \delta(\partial_\mu \phi^a) \right\}$$

Repeated indices summed. Looks like relativistically invariant but not related to relativity holds for N.R. systems

Next use $\delta(\partial_\mu \phi^a) = \partial_\mu (\delta\phi^a)$

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^a}\, \delta\phi^a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\, \delta\phi^a \right] - \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right) \right] \delta\phi^a \right\}$$

Second term is a total derivative can write as a surface integral. Usual variational principle $\delta\phi^a = 0$ at $t = \pm\infty$. Also assume vanishes as $|\vec{x}| \to \infty$. So then second term is zero
$\delta\phi^a(x)$ is arbitrary so

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi^a)} \right) - \partial_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_j \phi^a)} \right) = 0$$

Called the Euler Lagrange equations

Hamiltonian

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a}$$

$$H = \int d^3x \{ \pi^a \dot{\phi}^a - \mathcal{L} \} = \int d^3x\, \mathcal{H}$$

$$\mathcal{H} = \pi^a \dot{\phi}^a - \mathcal{L}$$

Example 1: Klein Gordon Field Theory

Consider a real scalar field. Scalar refers to how it transforms under Lorentz transformations. If Lagrange density is also a scalar you get a Lorentz invariant theory. We will say more about this later.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2)$$

$$= \frac{1}{2}(\partial_0 \phi \, \partial_0 \phi - (\vec{\nabla}\phi)^2 - m^2 \phi^2)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \, , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

Euler Lagrange equation

$$-m^2 \phi - \partial_\mu \partial^\mu \phi = 0$$

$$\frac{\partial^2}{\partial t^2}\phi - \vec{\nabla}^2 \phi + m^2 \phi = 0$$

Canonical momentum density

$$\pi(x) = \partial \mathcal{L} / \partial \dot{\phi} = \dot{\phi}$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2}(\pi^2 + (\vec{\nabla}\phi)^2 + m^2 \phi^2)$$

all terms positive

Example 2: Non Relativistic Field Theory

$$\mathcal{L} = i\phi^\dagger \dot{\phi} - \frac{(\vec{\nabla}\phi)^\dagger(\vec{\nabla}\phi)}{2m}$$

Now complex fields so might be confused. Lets break up into two real fields writing: $\phi = \phi_1 + i\phi_2$

$$\mathcal{L} = i(\phi_1 - i\phi_2)(\dot{\phi}_1 + i\dot{\phi}_2) - \frac{(\vec{\nabla}\phi_1)^2}{2m} - \frac{(\vec{\nabla}\phi_2)^2}{2m}$$

total derivative

$$= \cancel{i(\dot{\phi}_1\phi_1 + \dot{\phi}_2\phi_2)} + (\phi_2\dot{\phi}_1 - \phi_1\dot{\phi}_2) - \frac{(\vec{\nabla}\phi_1)^2}{2m} - \frac{(\vec{\nabla}\phi_2)^2}{2m}$$

$$\partial\mathcal{L}/\partial\phi_1 = -\dot{\phi}_2, \quad \partial\mathcal{L}/\partial\dot{\phi}_1 = \phi_2, \quad \partial\mathcal{L}/\partial(\partial_j\phi_1) = -\partial_j\phi_1/m$$

$$\partial\mathcal{L}/\partial\phi_2 = \dot{\phi}_1, \quad \partial\mathcal{L}/\partial\dot{\phi}_2 = -\phi_1, \quad \partial\mathcal{L}/\partial(\partial_j\phi_2) = -\partial_j\phi_2/m$$

Euler Lagrange equation $0 = \frac{\partial\mathcal{L}}{\partial\phi_j} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_j}\right) - \vec{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\phi_j)}$

Apply $j = 1, 2$

$$-\dot{\phi}_2 - \dot{\phi}_2 + \vec{\nabla}^2\phi_1/m = 0 \qquad\qquad -\dot{\phi}_2 + \vec{\nabla}^2\phi_1/2m = 0 \quad \#1$$

$$\Longrightarrow$$

$$\dot{\phi}_1 + \dot{\phi}_1 + \vec{\nabla}^2\phi_2/m = 0 \qquad\qquad +\dot{\phi}_1 + \vec{\nabla}^2\phi_2/2m = 0 \quad \#2$$

$$\#1 + i\#2 \Longrightarrow \quad i\dot{\phi} + \frac{\vec{\nabla}^2\phi}{2m} = 0$$

Back to complex field

Same as before ϕ, ϕ^+ as independent & varying w.r.t each one

$$\frac{\partial \mathcal{L}}{\partial \phi^+} = i\dot{\phi} \quad , \quad \frac{\partial \mathcal{L}}{\partial (\vec{\nabla}\phi^+)} = -\frac{\vec{\nabla}\phi}{2m}$$

$$0 = i\dot{\phi} + \frac{\vec{\nabla}^2\phi}{2m}$$

So its simpler just to view ϕ^+, ϕ as independent. Suppose we had varied wrt ϕ

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = i\phi^+ \quad , \quad \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} = -\frac{(\vec{\nabla}\phi)^+}{2m}$$

$$-i\dot{\phi}^+ - \frac{\vec{\nabla}^2\phi^+}{2m} = 0$$

again this is correct just "+" of above equation