A Fact From QM

Heisenberg Operators obey the classical equations of motion

\[ H = \frac{p^2}{2m} + V(x) \]

\[ I = \frac{x^2}{2m} - V(x) \]

Eqs of motion

\[ \dot{x} = \frac{p}{m} = [x, H]_\text{HS} \]

\[ \dot{p} = -\frac{dV}{dx} = [p, H]_\text{HS} \]

Quantum mechanics also leads

\[ i \dot{x}_t = [x_t, H] \]

\[ i \dot{p}_t = [p_t, H] \]

\[ [x_t, p_t] = i \]

\[ i \dot{x}_t = \left[ x_t, \frac{p_t^2}{2m} + V(x_t) \right] = \frac{2}{2m} [x_t, p_t] p_t = \frac{2i}{2m} \]

\[ x_t = \frac{p_t}{m} \]

\[ i \dot{p}_t = [p_t, \frac{p_t^2}{2m} + V(x_t)] \]
\[ V(x^\mu) = \sum_{n=0}^{\infty} \frac{V_n x^\mu}{n!} \]

\[ Lp_{\mu} x^{\mu} = n x^{\mu-1} \]

\[ \Rightarrow p_{\mu} x^{\mu} = -i n x^{\mu-1} \]

So

\[ Lp_{\mu} V(x^\mu) = -i \frac{dV(x^\mu)}{dx^\mu} \]

\[ x_{\mu} = \frac{p_{\mu}}{m} \]

\[ \Rightarrow x_{\mu} + \frac{dV(x^\mu)}{dx^\mu} = 0 \]

\[ p_{\mu} = -i \frac{dV(x^\mu)}{dx} \]

Continuum Field Theory in Classical + Quantum Physics

Suppose we have a canonical coordinate \( x^\mu \) and a momentum \( p(x) \). The Lagrange function is then given by:

\[ L = \int d^4x \mathcal{L}(\phi(x), \phi'(x)) \]

for simplicity, no gradients

\[ L = \sum_{\xi} \alpha^3 \mathcal{L}(\phi(x_i), \phi'(x_i)) \]

\( \phi(x_0) \equiv \phi_i \) canonical coordinate

\[ p_i = \frac{\partial L}{\partial \phi_i} = \alpha^3 \frac{\partial \mathcal{L}}{\partial \phi_i} \]

\[ p(x_i) \]

\[ H = \sum_{\xi} \mathcal{H}(x_i) \phi'(x_i) - L \]
Define canonical momentum density
\[ \pi(x) = p(x) / \alpha^3 \]
\[ H = \sum_i \alpha^3 \pi(x_i) \phi'(x_i) - L \]
\[ \pi(x) \rightarrow \partial x / \partial \phi' \]
Recall \[ [p_i, q_j] = -i \delta_{ij} \]
\[ \rightarrow [\pi(x_i), \phi(x_j)] = -i \alpha^3 \delta_{ij} = -i \delta^3(x_i - x_j) \]
Canonical commutator relation in-field-theory
\[ [\pi(x), \phi(y)] = -i \delta^3(x - y) \]
\[ L[\pi(x), \pi(y)] = [\phi(x), \phi(y)] = 0 \]
What is Lagrange density for NR free-field? \[ L = \int d^3x \left\{ \frac{i}{2} \phi^+(x) \phi'(x) + \phi^+(x) \partial^2 / \partial x^2 \phi(x) \right\} \]
\[ \pi(x) = \frac{\partial L}{\partial \phi'(x)} = i \phi^+(x) \]
\[ H = \int d^3x \pi(x) \phi'(x) - L = -\int d^3x \phi^+(x) \partial^2 / \partial x^2 \phi(x) \]
\[ [i \phi^+(x), \phi(x)] = -i \delta^3(x - y) \]
\[ \rightarrow [\phi^+(x), \phi(x)] = -\delta^3(x - y) \]
Relativistic Notation

\[ \eta_{\mu\nu} = \eta_{\mu'\nu'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Greek indices go over \( \{ \alpha, \beta, \gamma \} \) \( \text{roman over} \ \{ \mu, \nu, \rho \} \). Spaced indices are summed unless explicitly stated otherwise.

\[ T_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^1 dx^1 - dx^2 dx^2 - dx^3 dx^3 \]

\[ x^\mu = (x^0, \vec{x}) \]

In the space-time interval \( \eta \)

\[ x_i - x_i = \eta_{\mu\nu} x_i^\mu x_i^\nu = x_i^0 x_i^0 - \vec{x}_i \cdot \vec{x}_i \]

\( \text{4-vectors dot product} \)

\[ x_i \cdot x_i = x_i^\mu x_i^\mu = x_i^\mu x_i^\mu \]

\[ x_m = (t, \vec{x}) \]

A particle of mass \( m \) has four momentum

\[ p^\mu = (E, \vec{p}) \]

\[ p_\mu = \eta_{\mu\nu} p^\nu = E^2 - \vec{p}^2 = m^2 \]

\[ \Omega_m = \frac{p}{E} = (\vec{p}, \vec{q}) \]
Field Equations for Classical Field Theory

The action is the integral of Lagrangian over time. In local field theory, Lagrangian is integrated over space of Lagrangian densities.

\( S = \int dt d^4x \ L (\phi^a, \partial \phi^a) \)

depends on field time derivatives and Lorentz derivatives.

\( \phi^a = \phi^a(x) \). Minimization of the action.

\( 0 = \delta S = \int d^4x \left\{ \frac{\partial}{\partial \phi^a} \frac{\partial L}{\partial (\partial \phi^a)} + \frac{\partial}{\partial (\partial \phi^a)} \frac{\partial L}{\partial \phi^a} \right\} \delta (\phi^a) \)

\( = \int d^4x \left\{ \frac{\partial}{\partial \phi^a} \frac{\partial L}{\partial (\partial \phi^a)} + \frac{\partial}{\partial (\partial \phi^a)} \frac{\partial L}{\partial \phi^a} \right\} \delta (\phi^a) \)

Revised relations succeed. Looks like relativistically invariant, but not related to relativity holds for N.H. systems.
Next use $\delta (\tilde{L} \Phi^a) = \delta m \Phi^a$.

$$0 = \delta S = \int d^3 x \left[ \delta \left( \frac{d L}{d \Phi^a} \Phi^a + \delta m \left( \frac{d L}{d (\delta \Phi^a)} \delta \Phi^a \right) \right] - \left[ \delta m \left( \frac{d L}{d (\delta \Phi^a)} \right) \delta \Phi^a \right]$$

Let the last column vanish as a surface integral. Using a variation principle $\Phi^a = 0$ at $t = \pm \infty$. Theorem vanishes in $1^{st} - 2^{nd}$. So the second term gives $\delta \Phi^a$ arbitrary so

$$\frac{d L}{d \Phi^a} - \delta m \left( \frac{d L}{d (\delta \Phi^a)} \right) = 0$$

$$\frac{d L}{d \Phi^a} - \delta t \left( \frac{d L}{d (\delta \Phi^a)} \right) - \delta \Phi^a \left( \frac{d L}{d (\delta \Phi^a)} \right) = 0$$

Called the Euler-Lagrange equations.

**Hamiltonian**

$$H^a = \frac{d L}{d \dot{\Phi}^a}$$

$$H = \int d^3 x \left( \frac{d L}{d \dot{\Phi}^a} - L \right) = \int d^3 x \mathcal{H}$$

$$\mathcal{H} = H^a \dot{\Phi}^a - L$$
Example: Klein Gordon Field Theory

Consider a real scalar field. Scalar fields now transform under Lorentz transformations. If Lagrangian density is also scalar, you get a Lorentz invariant theory. We will say more about this later.

\[ L = \pm (\partial \phi \partial \phi - m^2 \phi^2) \]

\[ = \pm \left( \partial_\mu \phi \partial^\mu \phi - (\partial \phi)^2 - m^2 \phi^2 \right) \]

\[ \frac{\partial L}{\partial \phi} \]

Euler-Lagrange equation

\[ -m^2 \phi - \partial_\mu \partial^\mu \phi = 0 \]

\[ \frac{\partial L}{\partial \phi} \]

\[ \frac{\partial L}{\partial \phi} = \rho \phi \]

Canonical momentum density

\[ \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \phi \]

\[ \mathcal{H} = \pi \dot{\phi} - L = \frac{1}{2} \phi^2 + \frac{1}{2} \left( \partial \phi \right)^2 + \frac{1}{2} m^2 \phi^2 \]

\[ = \frac{1}{2} (\pi^2 + (\partial \phi)^2 + m^2 \phi^2) \quad \text{all terms positive} \]
Example 2: Non-Relativistic Field Theory

\[ z = c \dot{\phi} + \dot{\phi} + \frac{(\nabla \phi)(\nabla \phi)}{2m} \]

Now consider fields as complex. Let's break it into two real fields: \( \phi = \phi_1 + i \phi_2 \)

\[ z = i(\phi_1 - i \phi_2)(\phi_1 + i \phi_2) = \frac{(\nabla \phi_1)^2}{2m} - \frac{(\nabla \phi_2)^2}{2m} \]

\( \partial / \partial \phi_1 = - \phi_2 \), \( \partial / \partial \phi_2 = \phi_1 \), \( \partial / \partial \nabla \phi_1 = - \phi_2 / m \)

\( \partial / \partial \phi_1 = \phi_1 \), \( \partial / \partial \phi_2 = - \phi_2 \), \( \partial / \partial \nabla \phi_2 = - \phi_1 / m \)

Euler-Lagrange equation \( 0 = \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial \phi_1} \left( \frac{\partial L}{\partial (\nabla \phi)} \right) - \frac{\partial}{\partial \phi_2} \left( \frac{\partial L}{\partial (\nabla \phi)} \right) \)

Apply \( j = 1, 2 \)

\( - \phi_2 - \phi_2 + \nabla^2 \phi_1 / m = 0 \)

\( - \phi_1 + \phi_1 + \nabla^2 \phi_2 / m = 0 \)

(\#1 + \#2 = \( \frac{i \phi}{2m} + \nabla^2 \phi = 0 \))

\( \frac{i \phi}{2m} + \nabla^2 \phi = 0 \)

Back to complex field
Same as before, \( \phi \), \( \phi^* \) as independent variables with

\[
\frac{\partial L}{\partial \phi} = i \phi^*, \quad \frac{\partial L}{\partial (\bar{\psi} \phi)} = -\frac{\bar{\nabla} \phi}{2m}
\]

\[
0 = i \phi^* + \frac{\bar{\nabla} \bar{\psi}}{2m}
\]

So its simpler just to view \( \phi^* \), \( \phi \) as independent. Suppose we had varied \( \bar{\psi} \phi \)

\[
\frac{\partial L}{\partial \psi} = i \phi^*, \quad \frac{\partial L}{\partial (\bar{\nabla} \phi)} = -\frac{(\bar{\nabla} \phi)^*}{2m}
\]

\[-i \phi^* - \frac{\bar{\nabla}^2 \phi^*}{2m} = 0
\]

again this is really just 11" of above equation.