Noether's Theorem

A relationship exists between symmetries and conservation laws. For example, conservation of energy is associated with the translation invariance of the system with respect to spatial translation invariance.

Suppose we consider a change in the fields

\[ \phi^a(x) \rightarrow \phi^a(x) + \alpha \Delta \phi^a(x) \]

with \( \alpha \) an infinitesimal parameter. \( \Delta \phi \) some deformation of the fields. If this is a symmetry, it doesn't change the equations of motion. This is true if Lagrangian changes by \( \alpha \) modulo a total derivative term in action give a surface term which cancels out.

\[ L(x) \rightarrow L(x) + \alpha \int_{\Sigma} \left( \Delta \phi(x) \right) \]

might require some terms in action

\[ L \rightarrow L + \frac{\partial L}{\partial \phi^a} (\alpha \Delta \phi^a) + \frac{\partial L}{\partial (\partial \phi^a)} \partial_m (\alpha \Delta \phi^a) \]

\[ \Rightarrow \partial_m \left[ L^\Sigma \right] = \partial_m \left[ \frac{\partial L}{\partial (\partial \phi^a)} \Delta \phi^a \right] \]
So we have a current \( j^m \):

\[
j^m = \frac{\partial L}{\partial (\partial_0 \phi)} - \frac{\partial L}{\partial (\partial_0 \phi^*)}
\]

\[
\partial^m j^m = 0 \implies \frac{2}{a^+} j^0 + \nabla^m j^m = 0
\]

\[
Q = \int d^3x j^0
\]

\[
\frac{\partial Q}{\partial t} = 0
\]

Example 1: \( L = \frac{1}{2} m \dot{\phi}^2 \phi \) \( \phi \) real

Symmetry \( \phi \rightarrow \phi + \alpha \). Lagrange density invariant \( \implies J^m = 0 \)

\[
j^m = \frac{\partial L}{\partial (\partial_0 \phi)}
\]

Example 2: \( L = \partial_0 \phi^* \partial_0 \phi - m^2 \phi^* \phi \)

\( \phi - \phi^* = e^{i\phi} \phi^* = (\phi + i\phi^*) \) \( \dot{\phi} = i\phi \) \( \dot{\phi^*} = -i\phi^* \)

Lagrange density invariant \( \implies J^m = 0 \)

\[
j^m = \frac{\partial L}{\partial (\partial_0 \phi)} + \frac{\partial L}{\partial (\partial_0 \phi^*)}
\]

\[
= j^m \phi^* \phi - j^m \phi^* \phi^* = i (\phi \phi^* - \phi^* \phi)
\]
\[ \mathcal{L}_m = \dot{\phi}^2 - m^2 \phi^2 \]

\[ = \dot{\phi}^2 (m^2 \phi^2) = 0 \]

**Space-Time Translations**

So far we have considered only symmetries that are not connected to space-time. Noether's Theorem also works for space-time symmetries. Consider translations

\[ x'^m = x^m - a^m \]

\[ \phi'(x') = \phi(x) \Rightarrow \phi'(x-a) = \phi(x) \]

\[ \Rightarrow \phi'(x) = \phi(x+a) \]

\[ = \phi(x) + a^m \delta_{m}^k \phi(x) + \ldots \]

Suppose density is a scalar \[ \rho'(x') = \rho(x) \]

\[ \rho'(x') = \rho(x+a) = \rho(x) + a^m \partial_m \]

\[ = \rho(x) + a^m \partial_m \left( \rho \phi^2 \right) \]
Note for each $a^r\quad r = 0,1,2,3$ a symmetric tensor $J^{\mu}$ and the super symmetric case discussed above.

\[ J^{\mu} = \eta^{\mu \nu} \chi \]

\[ T^\nu = \frac{\partial}{\partial \phi} - \eta^\nu \chi \]

\[ \eta^\nu \chi \]

\[ T^\nu = 0 \]

\[ H = \int d^3x T^{00} = \int d^3x [ \frac{\partial \phi^0}{\partial \phi} - \chi] \]

\[ = \int d^3x [ \pi \phi^0 - \chi] \]

\[ P^c = \int d^3x T^{0i} = \int d^3x \frac{\partial \phi}{\partial \phi} \]

\[ P^c = -\int d^3x ( \frac{\partial \phi}{\partial \phi} ) \]

Note this is physical momentum not $\pi \phi^0$.
\[
\phi_{(x_1)} = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial t}
\]

\[
\phi_{(x_2)} = \frac{\partial}{\partial x_2} - i \frac{\partial}{\partial t}
\]

\[
\phi_{(x_3)} = \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial t}
\]

\[
L_0(x_1, t) = 0
\]

\[
L_0(x_2, t) = 0
\]

\[
L_0(x_3, t) = 0
\]

Heisenberg picture field

\[
\phi^\dagger_0 = \phi_0
\]

\[
H = \frac{\hbar}{2} \left( -\frac{\partial^2}{\partial x^2} + \frac{1}{4} x^4 - \frac{1}{2} \right)
\]

Deal with Heisenberg picture... To look at variances

Diagnosis: Quantizing M.R. Free Field Theory

\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} \phi
\]
When the commutation relation 1 b's

\[ [\phi(x, 0), \phi(\xi, 0)] = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \left[ \hat{b}(k) \hat{b}(\xi) \right] \]

\[ = \delta^3(x-\xi) \]

We have:

\[ [\hat{b}(x), \hat{b}(\xi)] = \frac{1}{(2\pi)^3} \delta^3(x-\xi) \]

\[ [\hat{b}(x), \hat{b}^+(x')] = [\hat{b}^+(x), \hat{b}(x')] = 0 \]

From Hamilton's proof

\[ H = \int \frac{d^3k}{(2\pi)^3} \frac{E_\xi}{\hbar} \hat{b}(k) \hat{b}^+(k) \]

So we state again \( |0\rangle = \hat{b}^+(x) |0\rangle \) and equivalent to another, same as before. But why call abelian this? Just label here.

\[ \bar{P} = -\int d^3x \frac{\partial}{\partial x} \nabla \phi \]

\[ = -\int d^3x \frac{\partial}{\partial x} \nabla \phi(x) \]

\[ = -\int d^3x \int \frac{d^3k}{(2\pi)^3} e^{-i(k \cdot x)} \int \frac{d^3k}{(2\pi)^3} e^{i(k \cdot x)} \hat{b}(k) \hat{b}^+(k) \]

\[ = -\int d^3x \int \frac{d^3k}{(2\pi)^3} e^{-i(k \cdot x)} \int \frac{d^3k}{(2\pi)^3} e^{i(k \cdot x)} \hat{b}(k) \hat{b}^+(k) \]
\[ \int \frac{d^3k}{(2\pi)^3} \rho b^\dagger(k) b(k) \]

\[ \vec{P}\ket{1R} = \vec{t}\ket{1R} \]

so its momentum!
Lorentz Transformations

A Lorentz transformation of space-time coordinates is a linear transformation

\[ x' = \Lambda x \]

where the interval \( ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \) is invariant.

\[ ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu dx^\mu dx^\nu \]

\[ = ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \]

So we have

\[ \eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu = \eta_{\alpha\beta} \]

Let us write the equation in matrix form

\[ \mathbf{\eta} = \mathbf{\Lambda}^T \mathbf{\eta} \mathbf{\Lambda} \]

Taking the determinant

\[ \det \mathbf{\Lambda}^2 = 1 \Rightarrow \det \mathbf{\Lambda} = \pm 1 \]

Transformations with \( \det \mathbf{\Lambda} = 1 \) are called proper Lorentz transformations. They can be continuously connected to the identity. Since \( \eta^2 = 1 \)

\[ 1 = \mathbf{\eta}^T \mathbf{\eta} \]

\[ \Rightarrow \mathbf{\Lambda}^{-1} = \mathbf{\eta} \mathbf{\eta}^T \mathbf{\eta} \]

\[ \Rightarrow 1 = \mathbf{\eta} \mathbf{\Lambda}^{-1} \mathbf{\eta} \]

\[ \Rightarrow \mathbf{\eta} = \mathbf{\eta} \mathbf{\Lambda}^{-1} \mathbf{\eta} \]

\[ \mathbf{\eta} = \Lambda^\alpha_\mu \eta_{\alpha\beta} \Lambda^\beta_\nu \]
Consider a Lorentz transformation infinitesimally close to the identity transformation

\[ A^\alpha = \eta^\alpha + \omega^\alpha \]

\[ \left( \text{Note: } \eta^\alpha = \delta^\alpha_\nu \right) \]

Now we have

\[ \eta_{\alpha \nu} \eta^{\alpha \beta} = \eta_{\nu \beta} \]

\[ \eta_{\alpha \nu} \left( \eta^\alpha_\alpha + \omega^\alpha_\alpha \right) \left( \eta^\nu_\beta + \omega^\nu_\beta \right) = \eta_{\alpha \beta} \]

\[ \eta_{\alpha \beta} + \omega_{\beta \alpha} + \omega_{\alpha \beta} = \eta_{\alpha \beta} \]

\[ \omega_{\beta \alpha} = -\omega_{\alpha \beta} \quad \frac{16-4}{2} = 6 \text{ parameters} \]

\[ 3 \text{ rotations} \quad 3 \text{ boosts} \]

\[ \eta_{ij} \text{ rotations} \]

\[ \omega_{ij} \text{ boosts} \]

Now let's apply Nöller procedure to Lorentz transformation. Suppose a scalar field

\[ \phi'(x) = \phi(x) \]

\[ \phi(x) = \phi(\Lambda^{-1} x) = \phi(x^{\mu} - \omega^{\mu}_{\nu} x^{\nu}) \]

\[ = \phi(x) - \omega^{\mu}_{\nu} x^{\nu} \partial_{\nu} \phi \]

\[ - \phi(x) - \frac{\omega^{\alpha \beta} (\eta^{\alpha}_{\mu} x^{\mu} - \eta^{\alpha}_{\beta} x^{\beta}) \partial_{\nu} \phi}{2} \]

\[ \Delta \phi_{\alpha \beta} = (x_{\alpha} \partial_{\beta} \phi - x_{\beta} \partial_{\alpha} \phi) \]

Similarly, upon imposing closure, we can states \( 2'(x) = x(x) \).
\[ Z'(x) = Z(x) \]
\[ Z'(x) = Z(x) - \frac{\omega^2}{2} \ln \left[ \frac{\left( \eta^a x^a - \eta^b x^b \right)^2}{2} \right] \]
\[ \left( J^a \right)_{\mu} = \left( -\eta^a x^b - \eta^b x^a \right) X^\mu \]

So current current

\[ J^a_{\mu} = \left( x^a \partial_\mu \phi - x^b \partial_\mu 2\phi \right) \frac{1}{2} + \left( \eta^a x^b - \eta^b x^a \right) X^\mu \]

Recall our formula for stress tensor

\[ j^a_{\mu} = \left( x^a T^\beta_\mu - x^b T^\beta_\mu \right) \]

Conserved charges are Lorentz transformed

\[ Q_{\mu} = \int d^3x \ j^0_{\mu} \]

\[ 0 = \frac{dQ_{\mu}}{dt} = \{ Q_{\mu}, H \} + \frac{dQ_{\mu}}{dt} \]

For Q\mu explicit time dependence so last term does not vanish. This in QM \{ p_\beta, L \} and we do not have commutators vanishing. But for Q\mu

\[ \partial\phi / \partial t = 0 \Rightarrow \{ \phi, H \} = 0 \] and the quantum numbers associated with position, time are angle momentum and \( T_i \sim \epsilon_i k Q_k \).