Representations of the Lorentz Group

Fields that transform linearly under Lorentz transformations form what is called a representation of the group (or basis for a representation).

\[ \Phi_a'(x') = \sum_{b=1}^r M_{ab}(\lambda) \Phi_b(x) \]

(usually use summation convention)

Matrix for \( \Phi'(x') = M \Phi(x) \)

Two successive transformations:

\[ \Phi' = M(\lambda_1) \Phi, \quad \Phi'' = M(\lambda_2) \Phi' \]

\[ \Phi'' = M(\lambda_2) M(\lambda_1) \Phi = M(\lambda_2 \lambda_1) \Phi \]

So we have \( M(\lambda_2) M(\lambda_1) = M(\lambda_2 \lambda_1) \). Matrices have same composition rules as Lorentz transformations.

Call \( r \)-dimensional representation if it is impossible to find less than \( r \) linear combinations of the \( \Phi_a \) that transform amongst themselves under Lorentz transformations.

If we change basis \( \Phi_a = \sum_b c_{ab} \Phi_b \)
Then \( \Phi_i = \text{Mat} \Phi_0 \)

\( \text{Sac} \Phi_i = \text{Mat} \text{Sod} \Phi_0 \)

\( \Phi_i = \text{Sac}^{-1} \text{Mat} \text{Sod} \Phi_i \)

\[ \Rightarrow M(n) = S \overline{M(n)} S \]

Two representations \( \Phi_i \), \( \Phi_i \) are equivalent.
Wells sometimes call the matrices \( \Phi_i \) the representation:
\( M_i \), \( \overline{M_i} \) are equivalent representations of the group.

**Rotation Group and Its Representations: A Review**

Rotations are a subgroup of Lorentz transformations and you already know all about their representations from undergraduate QM. We'll review these here, before going on to full Lorentz groups.

Can take a representation of rotation groups as 3x3 matrices:

\[ X_{ij} = R_{jk} X_k \]

Rotations preserve the inner product or norm of vectors and angle between vectors.
\[ x', y' = x - y \]

\[ \Rightarrow R^t R = 1 \]

\( R \) is a 3x3 orthogonal matrix with \( \det R = +1 \). So rotation group is \( SO(3) \). Rotations are parameterized by the quantities, e.g. Euler angles.

For rotations about \( x \), once through angle \( \Theta' \)

\[ R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta' & \sin \Theta' \\ 0 & -\sin \Theta' & \cos \Theta' \end{bmatrix} \]

\[ \approx \text{small \ angle} \begin{bmatrix} 1 \\ \text{small} \\ \text{angle} \end{bmatrix} = 1 + i\Theta' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \]

Similarly, for rotations about the other axes

\[ R_2 = 1 + i\Theta^2 \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \]

\[ R_3 = 1 + i\Theta^3 \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \]

Matrices

\[ J^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad J^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad J^3 = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \]

are called generators in 3-dim representation.
They are Hermitian traceless matrices. Obey commutation relations

\[
[J^i, J^j] = i \varepsilon^{ijk} J^k \quad (\varepsilon^{123} = 1)
\]

Eq. \[ J^i J^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}
\]

\[ J^2 J^1 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[ [J^i, J^2] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i J^3 \quad \square
\]

An arbitrary 3x3 rotation matrix can be made from composing infinitesimal transformations. One useful form is the infinitesimal

\[ R = \exp \left( \sum_{k=1}^{3} i \Theta^k J^k \right) \]

Suppose scalar field \( \phi \) rotate relations

\[ \phi'(\tilde{x}) = \phi(x) \]

\[ \Rightarrow \phi'(\tilde{x}) = \phi(R^{-1} \tilde{x}) = \frac{1}{4} (1 + i \Theta^k \hat{J}^k) \phi(x) \]

This is a differential operator

Let's work out \( \hat{J} \) explicitly.
\[ R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \quad R_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta' \\ 0 & \theta' & 1 \end{bmatrix} \]

\[ R_1^{-1}(x, y, z) = (x', y', z') = \left( \frac{x}{z+\theta' y}, \frac{y-\theta' z}{z+\theta' y}, z+\theta' y \right) \]

\[ \phi'(x') = \phi(x, y-\theta' z, z+\theta' y) \]

\[ = \phi(x) + \theta' \left( \frac{-z}{dy} + \frac{y}{dz} \right) \frac{\partial \phi}{\partial z} \]

\[ \hat{\mathbf{J}} = -i \begin{pmatrix} y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \\ z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \end{pmatrix} \]

Similarly,

\[ \hat{\mathbf{J}}^2 = -i \begin{pmatrix} z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \end{pmatrix}, \quad \hat{\mathbf{J}}^3 = \begin{pmatrix} x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \\ y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \end{pmatrix} \]

These are the angular momentum operators in QM:

\[ \hat{\mathbf{J}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}, \quad \hat{\mathbf{p}} = -i \hbar \nabla. \]

Differential operators

generally rotate with acting on fields, and these differential quasars satisfy the same commutation relations as matrices \( \mathbf{J} \).

\[ \{ \hat{J}^i, \hat{J}^j \} = i \epsilon^{ijk} \hat{J}^k \]
The algebra for generators $[J^i, J^j] = i\epsilon^{ijk} J^k$ doesn't depend on representation, it's a group property (SO(3)). We have found 3-dimensional representation for rotation group. Smallest representation is 2x2.

$$V = \exp \left[ i \frac{\sigma^k \sigma^k}{2} \right]$$

Since $$\left[ \frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i \epsilon^{ijk} \frac{\sigma^k}{2} \quad J^i = \sigma^i$$

Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Can express $V$ more explicitly, in terms of angle $\theta$. With

$$\theta^i = \hat{n}^i \quad \hat{n} \cdot \hat{n} = 1$$

$$V = \exp \left( i \frac{\sigma^k \sigma^k}{2} \right) = \exp \left( i \frac{\theta}{2} \hat{n} \cdot \sigma \right)$$

$$= \sum_{p=0}^{\infty} \frac{1}{(2p)!} \left( i \frac{\theta}{2} \hat{n} \cdot \sigma \right)^p = \sum_{p=0}^{\infty} \frac{1}{(2p)!} \left( \frac{1}{2} \right)^{2p} \theta^{2p} (\sigma \cdot \hat{n})^{2p}$$

$$= \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left( \frac{1}{2} \right)^{2p+1} \theta^{2p+1} (\sigma \cdot \hat{n})^{2p+1}$$

Rot+
\[(\mathbf{\hat{c}} \cdot \mathbf{\hat{n}})^2 = \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} = \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \left( \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} + \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \right) = \frac{\mathbf{\hat{c}} \cdot \mathbf{\hat{n}} (\mathbf{\hat{c}} \cdot \mathbf{\hat{n}})}{2} \]

\[\left[ \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \right] = 0 \mathbf{\hat{c}} + \mathbf{\hat{c}} \mathbf{\hat{n}} = 2 \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \]

\[\Rightarrow (\mathbf{\hat{c}} \cdot \mathbf{\hat{n}})^2 = 1 \quad \text{2x2 identity matrix} \]

\[(\mathbf{\hat{c}} \cdot \mathbf{\hat{n}})^2 \mathbf{\hat{n}} = \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \]

So,

\[U = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} \left( \mathbf{\hat{n}} \right)^{2p} + i \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} \left( \mathbf{\hat{n}} \right)^{2p+1} \]

\[= \cos \left( \frac{\mathbf{\hat{c}} \cdot \mathbf{\hat{n}}}{2} \right) + i \mathbf{\hat{c}} \cdot \mathbf{\hat{n}} \sin \left( \frac{\mathbf{\hat{c}} \cdot \mathbf{\hat{n}}}{2} \right) \]

With \( i/ \mathbf{\hat{c}} = 2 \pi i \hbar \) and many directions \( \mathbf{\hat{n}} \), \( U^2 = -1 \)

We have a 2-component Pauli spinor. Sometimes called a multihole spin of rotating group.

At level 1 algebraic spin \( S_0 ) \), \( S_0 / 2 \) are the same (globally different). You already know everything about finite dimensional representations of the rotation group. They are states \( | j, m > \)

\[\mathbf{J}^2 \mid j, m > = j(j+1) \mid j, m > \quad \mathbf{J}^3 \mid j, m > = m \mid j, m > \]

From lowering and raising operators: \( \mathbf{J}^2 = \mathbf{J}^+ \mathbf{J}^- = \mathbf{I} \)

\[\mathbf{J}^+ \mid j, m > = \sqrt{j(j+1) - m (m+1)} \mid j, m+1 > \quad \text{demoted} \quad m \rightarrow \]

\[\mathbf{J}^- \mid j, m-1 > = \sqrt{j(j+1) - m (m+1)} \mid j, m > \quad \text{demoted} \quad m \rightarrow -j-1 \]