Special Cases

$(j,0)$, $\vec{B}=0$, $\vec{J}=i\vec{K}$

$(0,j)$, $\vec{A}=0$, $\vec{J}=-i\vec{K}$

Simplest of these

$(1/2,0)$, $\vec{J}=\vec{\sigma}/2$, $\vec{K}=-i\vec{\sigma}/2$ Type I Spinors (R)

$(0,1/2)$, $\vec{J}=\vec{\sigma}/2$, $\vec{K}=i\vec{\sigma}/2$ Type II Spinors (L)

For type I

$$D_R(\Lambda)= \exp\left(i\vec{\Theta}\cdot\vec{J}+i\vec{\Phi}\cdot\vec{K}\right)=\exp\left[\frac{i\vec{\sigma}}{2}\left(\vec{\Theta}-i\vec{\Phi}\right)\right]$$

Under a Lorentz transformation

$$\Psi_R \to \Psi_R' = D_R(\Lambda)\Psi_R$$

For type II spinors

$$D_L(\Lambda)= \exp\left(i\vec{\Theta}\cdot\vec{J}+i\vec{\Phi}\cdot\vec{K}\right)=\exp\left[\frac{i\vec{\sigma}}{2}\left(\vec{\Theta}+i\vec{\Phi}\right)\right]$$

$$\Psi_L \to \Psi_L' = D_L(\Lambda)\Psi_L$$

Note there does not exist an S such that for every Λ $D_R(\Lambda)= S D_L(\Lambda) S^{-1}$. Hence representations $(1/2,0)$ and $(0,1/2)$ are not equivalent representations

But there is still a useful relationship between these two

$$D_R(\Lambda) = \epsilon D_L(\Lambda)^* \epsilon^{-1}$$

where $\epsilon = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\epsilon^2 = -1$, $\epsilon^{-1} = -\epsilon$

To verify this consider

$$\epsilon D_L^*(\Lambda)\epsilon^{-1} = \sigma^2 \exp\left(\frac{-i\vec{\sigma}^*}{2}(\vec{\Theta} - i\vec{\Phi})\right)\sigma^2$$

$$= \sigma^2 \sum_{n=0}^{\infty} \frac{[-i\vec{\sigma}^* \cdot (\vec{\Theta} - i\vec{\Phi})]^n}{2^n n!} \sigma^2$$

$$= \sum_{n=0}^{\infty} \frac{[-i(\sigma^2 \vec{\sigma}^* \sigma^2) \cdot (\vec{\Theta} - i\vec{\Phi})]^n}{2^n n!} \qquad \text{since } (\sigma^2)^2 = 1$$

But: $\sigma^2 \sigma^{j*} \sigma^2 = \sigma^2 \sigma^j \sigma^2 = -\sigma^2\sigma^2\sigma^j = -\sigma^j \qquad j = 1, 3$

$$\sigma^2 \sigma^{2*} \sigma^2 = -\sigma^2 \qquad j = 2$$

$$\epsilon D_L^*(\Lambda)\epsilon^{-1} = \sum_{n=0}^{\infty} \frac{[i\vec{\sigma} \cdot (\vec{\Theta} - i\vec{\Phi})]^n}{2^n n!}$$

$$= \exp\left(i\frac{\vec{\sigma}}{2} \cdot (\vec{\Theta} - i\vec{\Phi})\right) = D_R(\Lambda)$$

Scalars From Spinors

Consider the quantity $S_L = \psi_L^T \epsilon \psi_L = -i\psi_L^T \sigma^2 \psi_L$
Under a Lorentz transformation

$$S_L \to S_L' = -i\psi_L'^T\sigma^2\psi_L' = -i\psi_L^T D_L^T(\Lambda)\sigma^2 D_L(\Lambda)\psi_L$$

$$= -i\psi_L^T\sigma^2\left(\sigma^2 D_L^T(\Lambda)\sigma^2 D_L(\Lambda)\right)\psi_L$$

But: $$\sigma^2 D_L^T(\Lambda)\sigma^2 = \sigma^2\exp\left(\frac{i\vec{\sigma}^T}{2}(\vec{\theta} + i\vec{\phi})\right)\sigma^2$$

$$= \exp\left(\frac{i\sigma^2\vec{\sigma}^T\sigma^2}{2}(\vec{\theta}+i\vec{\phi})\right)$$

$$\sigma^{2T} = -\sigma^2 \quad , \quad \sigma^{jT} = \sigma^j \quad j=1,3$$

$$\sigma^2\vec{\sigma}^T\sigma^2 = -\vec{\sigma}$$

So, $$\sigma^2 D_L^T(\Lambda)\sigma^2 = \exp\left(\frac{-i\vec{\sigma}}{2}(\vec{\theta}+i\vec{\phi})\right) = D_L^{-1}(\Lambda)$$

$$S_L' = -i\psi_L^T\sigma^2 \cancel{D_L(\Lambda)^{-1} D_L(\Lambda)}\psi_L = S_L$$

Four Vectors From Spinors

We can also combine two spinors to a four vector. Lets focus on boosts. Recall that under boosts four vectors transform as

$$V \to V' = V + i\vec{\Phi}\cdot\vec{K}V$$

$$K^1 = -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad K^2 = -i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}, \quad K^3 = -i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

So

$$V^{0\prime} = V^0 + \vec{\phi}\cdot\vec{V}$$

$$V^{\prime j} = V^j + \phi^j V^0$$

Lets consider effects of a boost on $\psi_L^\dagger \psi_L$

$$(\psi_L^\dagger \psi_L)' = \psi_L^\dagger D_L(\Lambda)^\dagger D_L(\Lambda)\psi_L$$

$$\underset{\substack{\text{boost}}}{=} \psi_L \exp\left(-\frac{\vec{\sigma}\cdot\vec{\phi}}{2}\right)\exp\left(-\frac{\vec{\sigma}\cdot\vec{\phi}}{2}\right)\psi_L$$

$$\underset{\substack{\text{infinitesimal}\\ \text{boost}}}{=} \psi_L^\dagger (1 - \vec{\sigma}\cdot\vec{\phi})\psi_L = \psi_L^\dagger\psi_L - \phi^j \psi_L^\dagger \sigma^j \psi_L$$

Suggests four vector $V^\mu = (\psi_L^\dagger \psi_L, -\psi_L^\dagger \sigma^j \psi_L)$

We need to do one more calculation to do this

$$(\psi_L^\dagger \sigma^k \psi_L)' \simeq \psi_L^\dagger\left(1 - \frac{\vec{\sigma}\cdot\vec{\phi}}{2}\right)\sigma^k\left(1 - \frac{\vec{\sigma}\cdot\vec{\phi}}{2}\right)\psi_L$$

$$= \psi_L^\dagger \sigma^k \psi_L - \phi^j \psi_L^\dagger \left(\frac{\sigma^j\sigma^k + \sigma^k\sigma^j}{2}\right)\psi_L$$

$$= \psi_L^\dagger \sigma^k \psi_L - \phi^k \psi_L^\dagger \psi_L$$

So this works. Usually introduce notation $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and then $\psi_L \sigma^\mu \psi_L$ transforms as a four vector under Lorentz transformations.

A Lorentz Scalar Lagrange density is

$$\mathcal{L} = (\psi_L^\dagger i \bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_L^T \epsilon \psi_L)$$

need i for Hermitian Hamiltonian

Scalars + Spinors From Four Component Spinors

Parity takes a boost with velocity $\vec{V}$ into a boost with velocity $-\vec{V}$. In terms of generators we can think of parity taking, $\vec{K} \to -\vec{K}$, while $\vec{J} \to \vec{J}$. In other words parity interchanges $\vec{A}$ with $\vec{B}$, or equivalently ψ_L with ψ_R. So if the theory is parity invariant it must contain both ψ_L + ψ_R. Usually we put them together into a four component object

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \leftarrow \text{Dirac Four Component Spinor}$$

Under a Lorentz transformation

$$\psi \to \left(\begin{array}{c|c} \exp\left[\frac{i\vec{\sigma}}{2}(\vec{\theta}+i\vec{\phi})\right] & 0 \\ \hline 0 & \exp\left[\frac{i\vec{\sigma}}{2}(\vec{\theta}-i\vec{\phi})\right] \end{array}\right)\psi$$

$$= \left(\begin{array}{c|c} D_L(\Lambda) & 0 \\ \hline 0 & D_R(\Lambda) \end{array}\right)\psi \equiv \underbrace{D(\Lambda)}_{4\times 4 \text{ matrix}}\psi$$

Under parity

$$\psi \to \underbrace{\pm}_{\text{still some freedom}} \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)\psi$$

Lets introduce 4x4 "Gamma" matrices

$$\gamma^0 = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right), \quad \gamma^j = \left(\begin{array}{c|c} 0 & \sigma^j \\ \hline -\sigma^j & 0 \end{array}\right)$$

Satisfy anticommutation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Easy to construct invariants using these matrices since

$$D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

$$\gamma^0 D(\Lambda)^\dagger \gamma^0 = D(\Lambda)^{-1}$$

Lets check these

$$\gamma^0 D(\Lambda)^\dagger \gamma^0 = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)\left(\begin{array}{c|c} \exp\left[\frac{-i\vec{\sigma}}{2}(\vec{\theta}-i\vec{\phi})\right] & 0 \\ \hline 0 & \exp\left[\frac{-i\vec{\sigma}}{2}(\vec{\theta}+i\vec{\phi})\right] \end{array}\right)$$

$$\cdot \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} \exp\left[\frac{-i\vec{\sigma}}{2}(\vec{\theta}+i\vec{\phi})\right] & \\ \hline & \exp\left[\frac{-i\vec{\sigma}}{2}(\vec{\theta}-i\vec{\phi})\right] \end{array}\right) = D(\Lambda)^{-1} \checkmark$$

For the other relation lets just check it for infinitesimal boosts

$$\Lambda^0{}_\nu \gamma^\nu = \gamma^0 + \vec{\phi}\cdot\vec{\gamma}\ , \quad \Lambda^j{}_\nu \gamma^\nu = \gamma^j + \phi^j \gamma^0$$

$$D(\Lambda) = 1 - \vec{\phi}\cdot\begin{bmatrix} \vec{\sigma}/2 & 0 \\ 0 & -\vec{\sigma}/2 \end{bmatrix}$$

So

$$D(\Lambda)^{-1}\gamma^0 D(\Lambda) = \Lambda^0{}_\nu \gamma^\nu$$

$$\Rightarrow \left[\begin{bmatrix} \vec{\sigma}/2 & 0 \\ 0 & -\vec{\sigma}/2 \end{bmatrix}, \gamma^0\right] = \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$D(\Lambda)^{-1}\gamma^j D(\Lambda) = \phi^j \gamma^0$$

$$\left[\begin{bmatrix} \sigma^k/2 & 0 \\ 0 & -\frac{\sigma^k}{2} \end{bmatrix}, \gamma^j\right] = \delta^{kj}\gamma^0 = \delta^{jk}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

You can do matrix multiplication + see it works.
Now we have all the tools. $\psi^\dagger\gamma^0\psi$ is a Lorentz scalar

$$(\psi^\dagger\gamma^0\psi)' = \psi^\dagger D(\Lambda)^\dagger \gamma^0 D(\Lambda)\psi = \psi^\dagger\gamma^0(\gamma^0 D(\Lambda)^\dagger \gamma^0) D(\Lambda)\psi$$

$$= \psi^\dagger \gamma^0 D(\Lambda)^{-1} D(\Lambda)\psi = (\psi^\dagger\gamma^0\psi)$$

Usually introduce $\bar{\psi} = \psi^\dagger\gamma^0$ + write this Lorentz scalar as $\bar{\psi}\psi$. Similarly $\bar{\psi}\gamma^\mu\psi$ is a four vector since

$$(\bar{\psi}\gamma^\mu\psi)' = \psi^\dagger D(\Lambda)^\dagger \gamma^0\gamma^\mu D(\Lambda)\psi = \bar{\psi} D(\Lambda)^{-1}\gamma^\mu D(\Lambda)\psi$$

$$= \Lambda^\mu{}_\nu(\bar{\psi}\gamma^\nu\psi) \checkmark$$

Dirac Lagrange density is a Lorentz scalar

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \rightarrow \text{spin-1/2 parity invariant}$$

Equation of motion

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \leftarrow \left(\text{vary } \bar{\Psi}, \Psi \text{ independently to get this}\right)$$

More on Dirac Matrices

$$\gamma^0 = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right), \quad \gamma^i = \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline -\sigma^i & 0 \end{array}\right) \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Introduce

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

$$\gamma^{5\,2} = 1, \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{j\dagger} = -\gamma^j, \quad \gamma^{5\dagger} = \gamma^5$$

16 linearly independent 4x4 matrices. Can express any 4x4 matrix in terms of