Charge Conjugation

This discrete symmetry interchanges particles antiparticles

\[ C^{-1} \gamma(x) C = -i \gamma^2 \gamma^3(x) \]

\[ \gamma^5 = (0 \ 0i \ 0) \]

\[ \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ C^{-1} = C \text{ preserving parity} \]

So \( C^{-1} \) and \( \gamma_5 \) are even. Does a half-particle be

So \( C^{-1} = C \) preserving parity. Does a half-particle be

real matrix

\[ C^{-1} \gamma(x) C = \begin{pmatrix} -i \gamma^2 & -i \gamma^3 \end{pmatrix} \gamma(x) \]

\[ = \gamma(x) \]

\[ C^{-1} \gamma(x) C = \int \frac{d^4k}{(2\pi)^3} \sum_s \left[ C^{-1} \psi^s(k) \psi^s(k) e^{-ikx} \\
+ C^{-1} \phi^s(k) \phi^s(k) e^{ikx} \right] \]

\[ = \int \frac{d^4k}{(2\pi)^3} \sum_s \left[ \phi^s(k) e^{i(k^0x-x^2/2m)} e^{ikx} \\
+ \phi^s(k) e^{-i(k^0x-x^2/2m)} e^{-ikx} \right] \]

So this makes sense

\[ \sum_s C^{-1} \bar{\psi}^s(k) \psi^s(k) = \sum_s \bar{\phi}^s(k) e^{i(k^0x-x^2/2m)} \phi^s(k) \]

Consider using modified sense before
\[ V^{(2)}(\gamma) = -i \gamma^2 V^{(0)}(\gamma) \]
\[ V^{(0)}(\gamma) = -c \gamma^2 V^{(0)}(\gamma) \]

\[ C^{-1} A^{(1)}(\gamma) C = b^{(1)}(\gamma) \]
\[ C^{-1} A^{(2)}(\gamma) C = -b^{(2)}(\gamma) \]

On acting on the states:

\[ C \left\{ \bar{\beta}, 1 \right\}_F = \left\{ \bar{\beta}, 2 \right\}_F \]
\[ C \left\{ \bar{\beta}, 2 \right\}_F = -\left\{ \bar{\beta}, 2 \right\}_F \]

Let's consider the behavior of fermion bilinears under charge conjugation:

\[ C^{-1} \Phi \gamma C = \Phi^T (c \gamma^2)^+ \gamma^0 = \Phi^T (c \gamma^2) \gamma^0 \]
\[ C^{-1} \Phi \gamma^* C = \Phi^T (c \gamma^2) \gamma^0 (c \gamma^2) \gamma^* \]
\[ = \gamma^2 \gamma^0 \gamma^2 \]
\[ = + \gamma^0 \gamma^2 \]
\[ \text{fermion fields act on each} \]

Similalrly check the bilinear \[ C^{-1} \Phi \gamma^m \gamma_c = -i \Phi \gamma^m \gamma_c \]
\[ C^{-1}SC = \int d^4x \ e^{-i\mathbf{x} \cdot \mathbf{m}} \mathcal{F} \]
\[ = \int d^4x \ (2\pi \mathbf{r}) \ (i\gamma^\mu) \mathcal{F} - m \mathcal{F} \]
\[ = \int d^4x \ \mathcal{F} (i\gamma^\mu) \]
\[ \text{integrate by parts} \]
The U(1) Fermion Number Symmetry

\[ Z = \Psi(x_1) (i \gamma^\mu m) \Psi(x_1) \]

\[ \bar{\Psi} \gamma^\mu \psi(x_1) \] leaves Lagrange density invariant.

\[ j^\mu = \frac{\partial L}{\partial (2\gamma^\mu \psi)} \]

\[ Q = \int d^3x \, \bar{\Psi}(x_1) \gamma^\mu \psi(x_1) \]

Domain expansion

\[ Q = \int d^3x \, \sum_{E \in \mathbb{R}} (a^\dagger(E) a^\dagger(E) + b^\dagger(E) b^\dagger(E)) \]

Can always add constant & still neutral.

\[ Q = \int d^3x \, \sum_{E \in \mathbb{R}} (a^\dagger(E) a^\dagger(E) - b^\dagger(E) b^\dagger(E)) \]

\[ + \int d^3x \, \sum_{E \in \mathbb{R}} (a^\dagger(E) b^\dagger(E)) \]

\[ \text{constant set to zero so } Q(10) = 0. \text{ This procedure is sometimes denoted by } \hat{Q} \text{ to remind one that } \text{ a constant subtracted so that } \langle 0 | \hat{Q} | 10 \rangle = 0. \]

\( \hat{Q} \) is # fermionic - # anti fermionic.
The Electromagnetic Field

Lagrangian density for electromagnetic field is

\[ L = -\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} \]

\[ F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \]

Lagrangian density invariant with "gauge change."

\[ A_\alpha \rightarrow A_\alpha + \partial_\alpha \lambda \]

So not degree of freedom physical. For example, could choose \( A_\phi = 0 \) (axial gauge). This may cause problems when quantities are well defined (need to hold \( A_{\alpha \phi} \) as canonical coordinates). But \( \vec{A} \) 's still suffer by a gauge transformation represent the same class of freedom.

An classical electromagnetic remains gauge invariant by fixing a gauge: society invariant change (dual gauge).

\[ \partial_\alpha A^\alpha = 0 \]

Self for freedom is gauge transfer by \( \Delta \) that satisfies \( \partial_\alpha \Delta = 0 \). Lagrangian density

\[ L = -\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} = -\frac{1}{2} \partial_\alpha A_\beta F^{\alpha \beta} \]

E.g. 1: note

\[ \frac{\partial L}{\partial A_\alpha} = 0 \]

\[ \frac{\partial L}{\partial (\partial_\alpha A_\beta)} = -F^{\alpha \beta} \]

\[ \Rightarrow \partial_\alpha F^{\alpha \beta} = 0 \]
\[ \partial_x \partial^x A^\beta - \partial_\alpha \partial^\alpha A^\alpha = 0 \]

In Lorenz gauge

\[ \partial_\alpha \partial^\alpha A^\beta = 0 \]

Find canonical momenta

\[ \Pi^\beta = \frac{\partial L}{\partial \dot{A}^\beta} \quad \text{(Note notation $\Pi^\beta$ conjugate to $A^\beta$)} \]

\[ \left[ \Pi^\beta(x) , A^\alpha(x') \right] \gamma^{\alpha \beta} \delta^3(x-x') \]

\[ \gamma^{\alpha \beta} \quad \text{from raising index on $A$} \]

But \[ \Pi^\beta = \frac{\partial L}{\partial \dot{A}^\beta} = F^{\mu \beta} \Rightarrow \Pi^0 = 0 \]

Clearly not 0 (with commutation relations). Let invariance gauge fix \( \partial_\alpha A^\alpha = 0 \) in Lagrangian density

\[ L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} = -\frac{1}{4} F^{\mu \beta} F_{\mu \beta} \]

\[ = -\frac{1}{2} \left( \partial^\alpha A^\beta - \partial^\beta A^\alpha \right) \partial_\alpha A^\beta \quad \text{0 integral} \]

\[ S = \frac{1}{2} \int d^4x \left( \partial^\alpha A^\beta \partial_\alpha A^\beta - \partial^\beta A^\alpha \partial_\alpha A^\beta \right) \]

\[ L = -\frac{1}{2} \partial_\alpha A^\beta \partial^\beta A^\alpha \]

\[ \Pi^\beta = \frac{\partial L}{\partial \dot{A}^\beta} = -A^\beta \quad \text{not zero for $\Pi^0$ so proceed.} \]
\[ H = \int d^3x \left[ \mathcal{L}^\alpha A_\alpha - \mathcal{L} \right] \\
= \int d^3x \left[ -A^\beta A_\beta + \frac{i}{2} \partial^\alpha A_\beta \partial_\alpha A_\beta \right] \\
= \int d^3x \left[ -\frac{i}{2} A^\beta A_\beta - \frac{1}{2} \ddot{A}^\beta \ddot{A}_\beta \right] \\
P = \int d^3x (\mathcal{L}^\beta \ddot{A}_\beta) = \int d^3x A^\beta \ddot{A}_\beta \\
\text{Field eq } \Box A = 0 \\

Hence \( A^\alpha \) satisfies the field equation. To expand \( A^\alpha \) in terms of creation \& annihilation operators constructs basis of polarization vectors.

\[ \begin{align*} 
\epsilon^\alpha \quad & \text{four vector index} \\
\epsilon^\lambda \left[ k \right] \quad & \text{four momentum} \\
\epsilon_{\alpha \beta \gamma \delta} \quad & \text{Poincaré} \\
\end{align*} \]

We will choose a particular basis of polarization vectors to satisfy

\[ \eta_{\alpha \beta} \epsilon^\alpha (k) \epsilon^\beta (k) \left[ k \right] = \eta_{\alpha \beta}, \quad k^2 = 0 \]

Explicitly \( \eta^{\alpha \beta} = (1, 0) \)

\( e_0 (k) = n \)

\( e_3 (k) = \frac{\left( k - n \left( n \cdot k \right) \right)}{k \cdot n} \)

\( e_0 (k) \cdot e_0 (k) = \frac{1}{k^2} \left( k^2 - 2k \cdot n \cdot k \cdot n + \left( n \cdot k \right)^2 \right) = -1 \)

\( e_3 (k) \cdot e_3 (k) = \frac{1}{(n \cdot k)^2} \left( k^2 - 2k \cdot n \cdot k \cdot n + \left( n \cdot k \right)^2 \right) = -1 \)

\( e_{12} (k) = (0, \text{basis in space}, \text{basis in space}) \)

\( e_1 \)
\[ A^x (x) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=0,1,2,3} \frac{L_{\lambda} (k) e^{i(k \cdot x)}}{\mathcal{E}_k} \]

\[ \mathcal{E}_k = \sqrt{k^2 + m^2} \]

\[ \text{Commutation relation} \]

\[ [a^\dagger_x (k), a^j_x (k')] = -\eta^{jx} \delta_{k-k'} \]

\[ \lambda = \lambda' = 0 \]

\[ \text{and in terms of these modes} \]

\[ H = -\int \frac{d^3k}{(2\pi)^3} \mathcal{E}_k \sum_{\lambda, \lambda' \in \{0,1,2,3\}} \eta_{\lambda\lambda'} a^{\dagger x} (k) a^x (k) \]

\[ \bar{P} = -\int \frac{d^3k}{(2\pi)^3} \sum_{\lambda, \lambda'} \eta_{\lambda\lambda'} a^{\dagger x} (k) a^x (k) \]

Now we started with imposing condition

\[ \partial_x A^x = 0 \]

\[ \text{what to simplify action. The Quantum field} \]

\[ 0 = \partial_x A^x = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \left( -i \cdot a^\dagger (k) k \cdot e_\lambda (k) e^{-ik \cdot x} + \text{c.c.} \right) \]

\[ k \cdot e_0 (k) = k^0, \quad k \cdot e_3 (k) = \frac{k^2}{k \cdot n} \Rightarrow \frac{-k^0}{k \cdot n} = -k^0 \]

\[ k \cdot e_{\lambda x} = 0 \]
\[ a^0(k) - a^3(k) = 0 \quad \#
\]

We cannot ignore this because of different anticommutativity relations. Build up killed space \( a^4 + 10 \) in unusual way. Know because of gappy uncertainties too many states. View \# as a condition on the physical states \( \{ b \} \)

\[(a^0(k) - a^3(k)) \{ b \} = 0\]

So between physical states

\[ \langle b_1 | c | a^0 A^x | b_2 \rangle = 0 \]

Since:

\[ \langle b_1 | (a^0(k) - a^3(k)) | b_2 \rangle = \langle b_1 | a^0 - a^3 | b_2 \rangle = 0 \]

Presence of unphysical states or original killed space not a problem. Don't enter into physical ones. Acting on physical states, density medium \( a^0, a^3 \) cancel out of Hamiltonian momentum. They are expressed as sums over physical polarizations \( \lambda = 1,2 \).

\[ P^x = \int d^3k \sum_{\lambda=1}^{2} \frac{1}{(2\pi)^3} \left| a^\lambda(k) \right|^2 k^x \]
Let's examine this in more detail.

Building states $a^{-1} \chi$ and $a^{-1} \chi^\dagger$. For physical state $1b>^T$ form

$$1b>^T = (a^0(k)^+ + \beta a^3(k)^+) 1b>$$

For $1b>^T b$ be physical

$$0 = (a^0(k) - a^3(k)) 1b>$$

$$= \left( a^0(k) - a^3(k), a^{\dagger 0}(k) + \beta a^{\dagger 3}(k) \right) 1b>$$

$$= (2\pi)^3 d^3(k) [\beta - \beta] 1b>$$

$$\implies \beta = -\beta$$

So physical states built up by asking $a^{-1} \chi^0(k)^+ - a^{-1} \chi^3(k)^+$

First start with $10>^T + a^{-1} \text{ physical limit} a^{\dagger 0}(k) 10>^T, (a^0(k) - a^3(k)) 10>$. Label state satisfies physical condition but doesn't contribute to energy momentum. In fact it doesn't contribute anything except only because of remaining gauge invariance in Lorentz gauge. They state have zero norm:

$$\left| (a^{\dagger 0}(k) - a^3(k)^+) 1b> \right|^2$$

$$= \langle b| (a^{\dagger 0}(k) - a^3(k)^+) (a^{\dagger 0}(k) - a^3(k)^+) 1b>$$

$$= \langle b| -1 + 1 1b> \delta^{(3)}(k) = 0$$

This state is physical but irrelevant.
Dirac Propagator & Photon Propagator

Dirac Eq

\((i\gamma^\mu \gamma^\nu - m^2)\psi(x) = 0\)

Dirac Greens function \(S_F(x-x')\)

\((i\gamma^\mu \gamma^\nu - m^2)S_F(x-x') = i\gamma^\nu \delta^{\mu}(x-x') S_{xx'}\)

\(\text{Ward}\)

\(S_F(x-x') = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)^\alpha \not{s}(p)}{\not{p}^2 - m^2 + i\epsilon} e^{-i p(x-x')}\)

\[(2\gamma^\mu - m^2)S_F(x-x') = i\int \frac{d^4 p}{(2\pi)^4} \frac{(\not{p} - m)(\not{p} + m)^\alpha \not{s}(p)}{\not{p}^2 - m^2 + i\epsilon} e^{-i p(x-x')}\]

\[= \int \frac{d^4 p}{(2\pi)^4} \frac{\not{p}^2 - m^2 + i\epsilon}{\not{p}^2 - m^2 + i\epsilon} \not{s}(p) e^{-i p(x-x')}\]

So b/c \(\not{s}(p) = \frac{1}{\not{p}^2 - m^2 + i\epsilon} \not{s}(p)\)

\(\not{s}(p)\) \text{ is Feynman b/c boundary condition.}

\(S_F(x-x') = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)^\alpha e^{-i p(x-x')}}{\not{p}^2 - m^2 + i\epsilon}\)

Can show std

\(S_{\alpha\beta}F(x-x') = \Theta(x^0 - y^0) \langle 0 | \gamma_\alpha(x) \gamma_\beta(y) | 10 \rangle - \Theta(y^0 - x^0) \langle 0 | \gamma_\beta(y) \gamma_\alpha(x) | 10 \rangle = \langle 0 | T \{ \gamma_\alpha(x), \gamma_\beta(y) \} | 10 \rangle\)
For the photon in empty space, the equation for the

\[ \Delta^2 A_{\mu} = 0 \]

Green's function \( G_{\mu\nu}(x-y) \) satisfies

\[ \Delta^2 G_{\mu\nu}(x-y) = i \delta^4(x-y) \delta_{\mu\nu} \]

\[ G_{\mu\nu}(x-y) = \frac{\gamma_{\mu} \gamma_{\nu} (-i \epsilon)}{\epsilon^2 - m^2 + i\epsilon} \]

Can check with mode expansion etc.

\[ D_{\mu\nu}(x-y) = \langle 0\mid T \{ A_{\mu}(x) A_{\nu}(y) \} \mid 10 \rangle \]

where \( T \{ A_{\mu}(x) A_{\nu}(y) \} = O(x^0-y^0) A_{\mu}(x) A_{\nu}(y) \)

\[ + O(y^0-x^0) A_{\nu}(y) A_{\mu}(x) \]