Ph 205a Problem Set 2

1. The first lecture discussed many-particle systems of bosons; problem 1 on the previous homework asked you to extend the results derived in lecture. This problem concerns an analogous treatment of fermions, for which $\Psi(\ldots x_1, \ldots x_j, \ldots, t) = -\Psi(\ldots x_j, \ldots x_1, \ldots, t)$. $\Psi$ can be expanded as

$$
\Psi(x_1, \ldots, x_N, t) = \sum_{q_1, \ldots, q_N} C(q_1, \ldots, q_N, t) \psi_{q_1}(x_1) \ldots \psi_{q_N}(x_N)
$$

$$
= \sum_{n_1, \ldots, n_N = 0} f(n_1, \ldots, n_N, t) \Phi_{n_1 \ldots n_N}(x_1, \ldots, x_N)
$$

where

$$
\Phi_{n_1 \ldots n_N}(x_1, \ldots, x_N) = \begin{pmatrix}
\psi_{q_1}(x_1) & \ldots & \psi_{q_N}(x_N) \\
\psi_{q_1}(x_1) & \ldots & \psi_{q_N}(x_N) \\
\vdots & \ddots & \vdots \\
\psi_{q_1}(x_1) & \ldots & \psi_{q_N}(x_N)
\end{pmatrix}
$$

$$
= \left( \frac{n_1! \ldots n_N!}{N!} \right)^{1/2} \begin{pmatrix}
\psi_{q_1}(x_1) & \ldots & \psi_{q_N}(x_N) \\
\psi_{q_1}(x_1) & \ldots & \psi_{q_N}(x_N) \\
\vdots & \ddots & \vdots \\
\psi_{q_1}(x_1) & \ldots & \psi_{q_N}(x_N)
\end{pmatrix}
$$

Just as in the bosonic case, we may write

$$
|\Psi(t)\rangle = \sum_{n_1, \ldots, n_N} f(n_1, \ldots, n_N; t) |n_1, \ldots, n_N\rangle
$$

The only difference here is that the occupation numbers $n_1, \ldots, n_N$ are restricted to the values 0 and 1; this is the Pauli exclusion principle. The Schrödinger equation for fermions is just like the one for bosons, except that various phase factors appear due to the antisymmetry of the wavefunction. For example, consider the kinetic-energy term in the Schrödinger equation in the particle basis:

$$
i \frac{\partial}{\partial t} C(q_1, \ldots, q_N, t) = \sum_{k=1}^{N} \sum_{Q} \langle q_k | [T] | Q \rangle C(q_1, \ldots, q_{k-1}, Q, q_{k+1}, \ldots, q_N, t) + \ldots
$$

We would like to reorder the quantum numbers $q_k, Q$ so that they are in the same sequence on each side of the equation. $q_k$ on the left has been replaced by $Q$ on the right; when we move $Q$ into its proper place in the order, a phase factor will arise. For values of $Q$ that should come before $q_k$, this factor is $(-1)^{n_{q_k+1} + n_{q_k+2} + \ldots + n_{q_k-1}}$; if $Q$ comes after $q_k$, the factor is $(-1)^{n_{q_k+1} + n_{q_k+2} + \ldots + n_{q_k-1}}$. When we change variables and go over to the $f$ coefficients, the phase factors will remain:

$$
i \frac{\partial}{\partial t} |\Psi(t)\rangle = \ldots + \sum_{n'_1, \ldots, n'_N} \sum_{i<j} \langle i | [T] | j \rangle f(n'_i, \ldots, n'_j, \ldots, t) (n'_j + 1)^{1/2} (n'_j)^{1/2} \delta_{n'_i, 0} \delta_{n'_j, 1}
$$

$$
\times (-1)^{n'_{i+1} + n'_{i+2} + \ldots + n'_{j-1} \ldots n'_i + 1 \ldots n'_j - 1} \ldots
$$

Creation and annihilation operators for fermions satisfy anticommutation relations

$$\{a_r, a^\dagger_s\} = a_r a^\dagger_s + a^\dagger_s a_r = \delta_{rs}
$$

$$\{a_r, a_s\} = \{a^\dagger_r, a^\dagger_s\} = 0
$$

and act on states in the occupation number basis as follows:

$$a_s |n_s\ldots\rangle = \begin{cases}
(-1)^{S_s(n_s)^{1/2}} |n_s - 1\ldots\rangle & \text{if } n_s = 1 \\
0 & \text{otherwise}
\end{cases}
$$

$$a^\dagger_s |n_s\ldots\rangle = \begin{cases}
(-1)^{S_s(n_s + 1)^{1/2}} |n_s + 1\ldots\rangle & \text{if } n_s = 0 \\
0 & \text{otherwise}
\end{cases}
$$

$$a^\dagger_s a_s |n_s\ldots\rangle = n_s |n_s\ldots\rangle \quad n_s = 0, 1
$$
where the phase factor $S_s$ is defined by

$$S_s = n_1 + n_2 + \ldots + n_{s-1}$$

Write the Schrödinger equation for fermions just as you did the one for bosons, in terms of $\langle ij | T | j \rangle$ and $\langle ij | V | k \ell \rangle$, carefully keeping track of all phase factors. Show that the Hamiltonian can be represented as $\hat{T} + \hat{V} = \sum_{ij} a_i^\dagger \langle i | T | j \rangle a_j + \sum_{ij\kappa\ell} a_i^\dagger a_j^\dagger a_i^\dagger a_j^\dagger \langle ij | V | k \ell \rangle a_{\kappa} a_{\ell}$. (Note the ordering of the last two annihilation operators.)

2. Consider a field theory with $N$ real scalar fields $\phi^a$, $a = 1, \ldots N$. The Lagrange density is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^a \partial^\mu \phi^a - m^2 \phi^a \phi^a) - \frac{\lambda}{4} (\phi^a \phi^a)^2$$

where repeated indices $a$ are summed over $1, \ldots N$.

a) What are the equations of motion?

b) Show that the transformations $\phi^a \rightarrow \phi'^a = \phi^a + \omega^{ab} \phi^b$ leave the Lagrangian density invariant provided that the infinitesimal parameters $\omega^{ab}$ satisfy

$$\omega^{ab} = -\omega^{ba}.$$ 

c) What are the conserved currents associated with these symmetries?

3. For Klein-Gordon theory, $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$, show that the stress tensor is

$$T^{\mu\nu} = \partial^\mu \phi \partial^{\nu} \phi - \frac{\eta^{\mu\nu}}{2} (\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2).$$