Ph 205a Problem Set 1

1. The Schrödinger Equation in occupation number basis is
   \[
i \frac{\partial}{\partial t} f(n_1, \ldots, n_\infty; t) = \sum_i \langle i|T|i \rangle n_i f(n_1, \ldots, n_i, \ldots, n_\infty; t) + \sum_{i \neq j} \langle i|T|j \rangle n_i^{1/2}(n_j + 1)^{1/2} f(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_\infty; t) + \text{potential terms}
   \]

   Find the potential terms. Express your answer in terms of \( \langle ij|V|kl \rangle \). Use the result to show that
   \[
   \hat{V} = \frac{1}{2} \sum_{ijkl} b_i^\dagger b_i^\dagger (ij)|V|kl b_k b_l \text{ is a representation of the potential acting on the occupation number basis } [n_1 \ldots n_\infty].
   \]

2. Show that the number operator \( \hat{N} = \int d^3x \phi^\dagger(x) \phi(x) \) commutes with
   \[
   \hat{H} = \hat{T} + \hat{V}
   \]
   where
   \[
   \hat{T} = \int d^3x \phi^\dagger(x) T(x) \phi(x)
   \]
   \[
   \hat{V} = \frac{1}{2} \int d^3x d^3y \phi^\dagger(x) \phi^\dagger(y) V(x, y) \phi(x) \phi(y)
   \]

3. Calculate the transition matrix element
   \[
   \langle k_1', k_2'|\hat{V}|k_1, k_2 \rangle
   \]
   where \( |k_a, k_b \rangle = b^\dagger(k_a) b^\dagger(k_b) |0 \rangle \) is the two-particle state, \( \hat{V} \) is given in problem 2, and \( V(x, y) = \frac{e^2}{|x-y|} \), with \( e \) a constant.

   Express your answer in terms of \( k_1', k_2', k_1, \) and \( k_2 \). One of the spatial integrals you must do is rather tricky, because it does not have a well-defined limit; a standard way of regulating it (i.e. giving it a meaningful value) is to insert a factor of \( e^{-\mu|x-y|} \) and then, after integrating, take the limit \( \mu \rightarrow 0 \).

4. a) Work out the commutators \( [\hat{N}, \phi(x)] \) and \( [\hat{H}, \phi(x)] \), with \( \hat{H} \) as in problem 2.
   b) Let \( |\Psi_N(t)\rangle \) satisfy \( \hat{N} |\Psi_N(t)\rangle = N |\Psi_N(t)\rangle \) and \( i \frac{d}{dt} |\Psi_N(t)\rangle = \hat{H} |\Psi_N(t)\rangle \), \( \langle \Psi_N(t)|\Psi_N(t)\rangle = 1 \).

   Using the results of part (a), show that the function
   \[
   \Psi_N(x_1, \ldots, x_N, t) \equiv \frac{1}{\sqrt{N!}} \langle 0|\phi(x_1) \ldots \phi(x_N)|\Psi_N(t)\rangle
   \]
   satisfies
   \[
   \int (\prod_{i=1}^N d^3x_i) |\Psi_N(x_1, \ldots, x_N, t)|^2 = 1
   \]
   \[
   i \frac{\partial}{\partial t} \Psi_N(x_1, \ldots, x_N, t) = \left[ \sum_{k=1}^N -\frac{1}{2m} \nabla_k^2 + \frac{1}{2} \sum_{k \neq l} V(x_k, x_l) \right] \Psi_N(x_1, \ldots, x_N, t)
   \]