1. State and prove Wick’s Theorem for anticommuting Fermi fields.

2. Show that the disconnected vacuum diagrams cancel out from the diagrammatic expansion of the two-point function

\[ \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle \]

in \( \lambda \phi^4 \) theory, up to and including terms of order \( \lambda^2 \).

3. a) Show that Wick’s Theorem can be written as

\[ T [ \phi_1(x_1) ... \phi_l(x_n) ] =: \exp \left[ \frac{1}{2} \int d^3x d^3y D_F(x - y) \frac{\delta}{\delta \phi_1(x)} \frac{\delta}{\delta \phi_1(y)} \right] \phi_1(x_1) ... \phi_1(x_n) : \]

where the functional derivative \( \frac{\delta}{\delta \phi_1(x)} \) satisfies

\[ \frac{\delta \phi_1(y)}{\delta \phi_1(x)} = \delta^4(x - y). \]

b) Consider the following “interacting” field theory:

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \rho(x) \phi(x) \]

where \( \rho(x) \) is a c-number function with \( \rho(x) \to 0 \) as \( |x| \to \infty \). Using the previous result, show that.

\[ U(\infty, -\infty) = T \exp \left[ -i \int_{-\infty}^{\infty} dt H_I(t) \right] \]

is given by

\[ U(\infty, -\infty) = \exp \left[ - \frac{1}{2} \int d^3x d^3y \rho(x) \rho(y) D_F(x - y) \right] \exp \left[ -i \int d^4z \rho(z) \phi_1(z) \right]. \]

c) Compute

\[ P(n) = \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} ... \frac{d^3k_n}{(2\pi)^3 2E_{k_n}} | \langle k_1 ... k_n | U(\infty, -\infty) | 0 \rangle |^2. \]

This is the probability of finding \( n \) particles in the far future starting with the vacuum in the far past. (Since \( \rho(x) \to 0 \) as \( t \to \pm \infty \), the initial/final states coincide with those of the free field theory. Show that \( n \) is Poisson distributed:

\[ P(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \]

with

\[ \lambda = \int \frac{d^3k}{(2\pi)^3 2E_k} | \tilde{\rho}(k, E_k) |^2. \]

Here

\[ \tilde{\rho}(k, E_k) = \int d^4x \rho(x) e^{ik \cdot x}, \]

with \( k^0 = E_k \).