Path Integral Quantization

Review of Quantum Mechanics

Consider a one-dimensional system with Hamiltonian:

\[ H = \frac{p^2}{2m} + V(x) \]

Suppose we want to calculate the QM amplitudes for a particle to propagate from the point \( x_a \) to point \( x_b \) in time \( T \). We call this \( U(x_a, x_b; T) \), in Schrödinger picture where operation are time independent.

\[ U(x_a, x_b; T) = \langle x_b | e^{-iHT} | x_a \rangle \]

In path integral formulation of QM this transition amplitude is

\[ U(x_a, x_b; T) = \int \mathcal{D}L(x(t)) e^{i S[x(t)]} \]

\( \mathcal{D}L(x(t)) \) is measure on integration over paths that connect \( x_a = x(0) \) to \( x_b = x(T) \) and \( S[x(t)] \) is the classical action for that path.

Note that...
\[
\frac{\partial}{\partial t} U(x_0, x_b; T) = \langle x_b | H | e^{-iHT} | x_0 \rangle \\
= \int dx \langle x_b | H | x \rangle \langle x | e^{-iHT} | x_0 \rangle \\
= \int dx \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \right) \delta(x-x_0) U(x, x_0; T) \\
= \left( -\frac{1}{2} \frac{\partial^2}{\partial x_b^2} + V(x_0) \right) U(x_0, x_0; T)
\]

So \( U \) satisfies the Schrödinger equation.

I want to show that above path integrals \\
do indeed give the quantum mechanical amplitude. To evaluate the path integral we need to define the measure \( D[x(t)] \). Try \\
brute force method by subdividing the problem \( D[x(0, T)] \) into thin strips of size \( \epsilon \)

\[
\int D[x(t)] = \frac{1}{C(\epsilon)} \int \frac{dx_1}{C(\epsilon)} \int \frac{dx_2}{C(\epsilon)} \ldots \int \frac{dx_{N-1}}{C(\epsilon)} \\
= \frac{1}{C(\epsilon)} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty} \frac{dx_k}{C(\epsilon)} \\
\text{One } C(\epsilon) \text{ for each slice (i.e. time interval)}
\]
Now $U(x_0, z; T - \epsilon)$ is related to $U(x_0, x_0; T)$ by

$$U(x_0, x_0; T) = \int_{-\infty}^{\infty} \frac{dz}{C(\epsilon)} \exp \left[ \frac{i m (x_0 - z)^2}{\epsilon} - i \epsilon V\left(\frac{x_0 + z}{2}\right) \right]$$

$$U(x_0, z; T - \epsilon)$$

Note

$$S[x(t)] = \int dt \left[ \frac{m \dot{x}^2}{2} - V(x(t)) \right]$$

$$= \sum_{j=0}^{N-1} \left[ \frac{m (x_{j+1} - x_j)^2}{2 \epsilon^2} - V(x_{j+1} + x_j) \right]$$

The integrand is smooth in $\epsilon$ except for kinetic term. So expand in $\epsilon$ terms that are smooth in $\epsilon$.

$$U(x_0, x_0; T) = \int_{-\infty}^{\infty} \frac{dz}{C(\epsilon)} \exp \left[ \frac{i m (x_0 - z)^2}{\epsilon} \right]$$

$$(1 - \epsilon \frac{\partial}{\partial T} - i \epsilon V(x_0)) \left(1 + (2 - x_0)^2 \frac{\partial}{\partial x_0} \right) U(x_0, x_0; T)$$

neglecting higher order terms in expansion.
Embedder $\tilde{z} = z - x_0 + \omega$

$$\int d\tilde{z} e^{-\frac{b\tilde{z}^2}{2}} = \sqrt{\frac{\pi}{b}} \int d\tilde{z} \tilde{z} e^{-\frac{b\tilde{z}^2}{2}} = 0$$

$$\int d\tilde{z} \tilde{z} e^{-\frac{b\tilde{z}^2}{2}} = \frac{1}{2b} \sqrt{\frac{\pi}{b}}$$

Hence

$$b = \left(\frac{-2\imath}{2\epsilon}\right)$$

$$U(x_0, x_0, T) = \frac{1}{\sqrt{2\pi\sigma}} \int \frac{d^2\epsilon}{\imath\pi} \left[ -\epsilon \frac{2\imath}{m \sigma} \right] U(x_0, x_0, T)$$

Now if $\epsilon \in (\epsilon) = \left[ \frac{-2\epsilon\imath}{m \sigma} \right]$ the above gives

$$0 = \left[ -\epsilon \frac{2\imath}{m \sigma} + \frac{\imath V(x_0)}{2m \sigma} \right] U(x_0, x_0, T)$$

$$0 = \left[ \frac{-\imath}{m \sigma} \frac{2\imath}{x_0^2} + \frac{\imath V(x_0)}{2m \sigma} \right] U(x_0, x_0, T)$$

So $U$ obeys the SE.

Now let's start with ordinary QM + derive this. Write

$$e^{-i\mathcal{H} T} = \sum_{n=0}^{\infty} e^{-i\mathcal{H} T} e^{-i\mathcal{H} E \sigma \frac{N-1}{N}}$$
Get dense like: \[ \langle x_{n+1} | e^{-iH\varepsilon} | x_n \rangle \]
\[ \underset{\varepsilon \to 0}{\to} \langle x_{n+1} | 1 - i\varepsilon H | x_n \rangle \]

Boundary value problem on well notation \( x_0 = a, \ x_N = b \).

Now

\[ \langle x_{n+1} | f(x) | x_n \rangle = f(x_n) \delta (x_{n+1} - x_n) = f(x_n + x_{n+1}) \delta (x_{n+1} - x_n) \]

Similarly

\[ \langle x_{n+1} | f(p) | x_n \rangle = \int dp_k \langle x_{n+1} | f(p) | p_k \rangle \]

\[ = \int \frac{dp_k}{2\pi} f(p_k) e^{i p_k (x_{n+1} - x_n)} \]

So

\[ H = H(x, p) = p^2/(2m) + V(x) \]

\[ \langle x_{n+1} | H | x_n \rangle = \int \frac{dp_k}{2\pi} H \left( \frac{x_{n+1} + x_n}{2} , p_k \right) e^{i p_k (x_{n+1} - x_n)} \]

\[ \mathcal{Z}(x_0, x_N ; t) = \frac{i}{h} \int dx_k \int dp_k \exp \left\{ i \left[ \sum_k p_k (x_{k+1} - x_k) \right] - \varepsilon H \left( \frac{x_{k+1} + x_k}{2} , p_k \right) \right\} \]

\[ = \frac{1}{h} \int dx_k \int dp_k \exp \left\{ i \left[ \sum_k p_k (x_{k+1} - x_k) \right] \right\} \exp \left\{ - \varepsilon H \left( \frac{x_{k+1} + x_k}{2} , p_k \right) \right\} \]

Now do explicitly the \( N \) \( p_k \) integrals.
\[ \int \frac{d\rho k}{2\pi} \exp \left[ i \left( \rho k \left( x_{k+1} - x_k \right) - \frac{\epsilon \rho k^2}{2m} \right) \right] \]

\[ = \int \frac{d\rho k}{2\pi} \exp \left( i \left[ -\frac{\epsilon}{2m} \right] \left( \rho k - \frac{(x_{k+1} - x_k) m}{\epsilon} \right)^2 \right) \]

\[ \cdot \exp \left[ i m \frac{(x_{k+1} - x_k)^2}{2\epsilon} \right] \]

\[ = \left[ \frac{\pi^{n/2} \epsilon^{-n/2}}{2^n} \right] \exp \left[ \frac{-i m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} \right] \]

\[ = \left[ \frac{\pi^{n/2} \epsilon^{-n/2}}{2^n} \right] \exp \left[ \frac{-i m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} \right] \]

So

\[ w(x_0, x_{n+1}; T) = \frac{1}{C} \prod_{k=1}^{n} \int \frac{d\rho k}{C} \]

\[ \cdot \exp \left\{ i \epsilon \left[ \frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon^2} - \frac{1}{2m} \frac{(x_{k+1} - x_k)^2}{\epsilon^2} \right] \right\} \]

\[ = \int_{D[L(x(T))]} e^{SL(x(T))} \]

Note if no potential can be integrated explicitly e.g.

\[ w(x_0, x_{n+1}; T) = \left( \frac{2\pi i T}{m} \right)^{-n/2} \exp \left( \frac{i m (x_{n+1} - x_0)^2}{2T} \right) \]

In quantum field theory, field of scalar

With coordinates, no formal...
\[ \langle \phi_0 (x) | e^{-iHT} | \phi_0 (x) \rangle = \int D[\phi(x)] \exp \left[ i \int dt \int d^3x \mathcal{L} (\phi(t)) \right] \]

\[ \phi(0, x) = \phi_0 (x) \]
\[ \phi(T, x) = \phi_0 (x) \]
\[ \mathcal{L} (\phi(t)) = \frac{1}{2} \partial \phi \bar{\partial} \phi - V(\phi) \]

**Correlation Functions**

To make contact with our previous work we need a functional formula for computing expectation values of four ordered products. Start with

\[ \int D[\phi(x)] \exp \left[ i \int d^3x \mathcal{L}(\phi) \right] \phi(x_1) \phi(x_2) \]

with b.c. \( \phi(-T, x) = \phi_0 (x) \), \( \phi(T, x) = \phi_0 (x) \). We need to relate this to

\[ \langle 21 \mid T(\phi_1(x_1), \phi_1(x_2)) \mid 22 \rangle \]

where \( \phi_1(x) \) is Heisenberg field. Denote Schrödinger

field by \( \phi \), superscript "S".

We will use

\[ \int D[\phi(x)] = \int d[\phi_1(x)] \int d[\phi_2(x)] \]

\[ \int D[\phi(x)] \]
\[ \phi(x_1, \bar{x}) = \phi_1 (\bar{x}) \]
\[ \phi(x_2, \bar{x}) = \phi_2 (\bar{x}) \]

\( \phi_1 \) , \( \phi_2 \) become \( \phi_1 (\bar{x}), \phi_2 (\bar{x}) \).