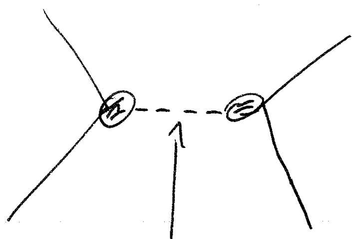
# Resonances and Widths of Unstable Particles

One way to search for new particles is to scatter other particles with enough energy to make the new particle. Schematic Feynman diagram:

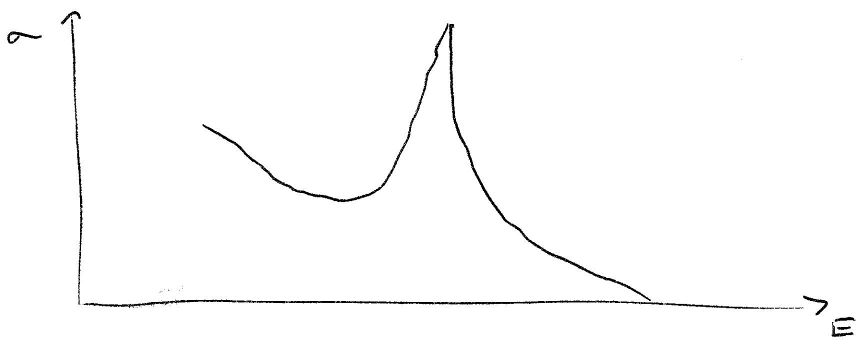


Propagator $\dfrac{i}{p^2 - M_X^2}$

Scattering cross-section $\propto \dfrac{1}{|p^2 - M_X^2|^2}$

If we scatter our particles so that $p^2 = M_X^2$ then $\sigma = \infty$ ?? Oh oh.

Cross-section for $e^+e^-$ scattering near mass of Z looks like




You do get a big bump, but it's not infinite.

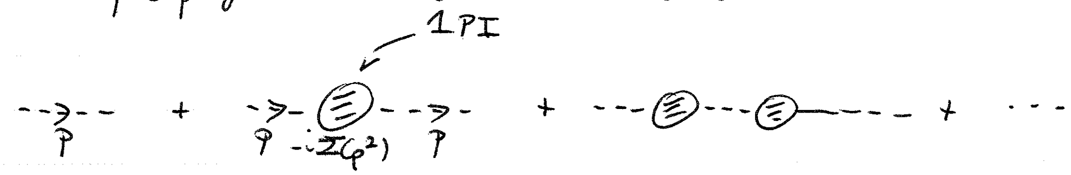
What's happening here is that the finite width of the $Z$ cuts off the infinity: $Z$ propagator is modified. $Z$ is a resonance.

We want to understand how this comes about. Consider a toy theory with heavy unstable pscalar $\chi$ and a stable $\varphi$.

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}M_\chi^2\chi^2 + \frac{h}{2}\chi\varphi^2.$$

If $M_\chi \geq 2m_\varphi$, then $\chi \to \varphi\varphi$ is allowed.

$\chi$ propagator and corrections:

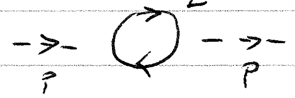



$$= \frac{i}{p^2 - M_\chi^2} + \frac{i}{p^2 - M_\chi^2}\left[-i\Sigma(p^2)\right]\frac{i}{p^2 - M_\chi^2} + \frac{i}{p^2 - M_\chi^2}\left[-i\Sigma\right]\frac{i}{p^2 - M_\chi^2}\left[-i\Sigma\right]\frac{i}{p^2 - M_\chi^2} + \cdots$$

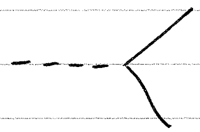
$$= \frac{i}{p^2 - M_\chi^2 - \Sigma(p^2)} \qquad \text{(Usual thing)}$$

At order $\cancel{g}h^2$, one graph contributes to $\Sigma$.

$\varphi$ particles.

$$-i\Sigma(p^2) =$$ 

Compute this graph. Need Feynman rule

 $= ih$

$$-i\Sigma(p^2) = \frac{(ih)^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{(p+q)^2 - m^2 + i\varepsilon} \frac{i}{q^2 - m^2 + i\varepsilon}$$

$$= \frac{h^2}{2} \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 + 2x p \cdot q + x p^2 - m^2 + i\varepsilon]^2}$$

$$\text{Denominator} = (q + xp)^2 - x^2p^2 + xp^2 - m^2 + i\varepsilon$$

$$= \ell^2 + x(1-x)p^2 - m^2 + i\varepsilon$$

$$-i\Sigma = \frac{h^2}{2} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 + x(1-x)p^2 - m^2 + i\varepsilon]^2}$$

Wick rotate $\ell^0 \to i\ell^0$

$$-i\Sigma = \frac{ih^2}{2} \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{[\ell_E^2 + m^2 - x(1-x)p^2 - i\varepsilon]^2}$$

The loop integral is, in dim reg $n = 4 - \varepsilon$

$$\int \frac{d^n l_E}{(2\pi)^n} \frac{1}{[l_E^2 + m^2 - x(1-x)p^2 - i\varepsilon]^2}$$

$$= \frac{1}{(2\pi)^2} \frac{\Gamma(\varepsilon/2)}{(4\pi)^{-\varepsilon/2}} \left(m^2 - x(1-x)p^2 - i\varepsilon\right)^{-\varepsilon/2}$$

$$= \frac{1}{(4\pi)^2}\left(\frac{2}{\varepsilon} - \gamma\right)\left(1 + \frac{\varepsilon}{2}\log 4\pi\right)\left(1 - \frac{\varepsilon}{2}\log(m^2 - x(1-x)p^2 - i\varepsilon)\right)$$

$$= \frac{1}{(4\pi)^2}\Big[\underbrace{\frac{2}{\varepsilon} - \gamma + \log 4\pi}_{\text{Drop these in } \overline{MS}} - \log(m^2 - x(1-x)p^2 - i\varepsilon)\Big]$$

$$\overset{\overline{MS}}{=} \frac{-1}{(4\pi)^2} \log\left(m^2 - x(1-x)p^2 - i\varepsilon\right)$$

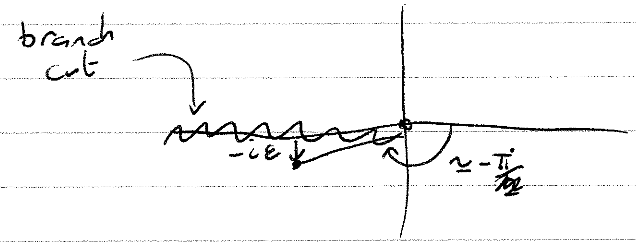
$$\therefore \Sigma(p^2) = \frac{h^2}{2(4\pi)^2}\int_0^1 dx \log\left(m^2 - x(1-x)p^2 - i\varepsilon\right).$$

Now do the $x$ integral, whole thing is just some mass renormalisation?

No! If $m^2 < x(1-x)p^2$, then the log has an imaginary part!

Compute imaginary part.

When $p^2 x(1-x) > m^2$, the imaginary part of the log is



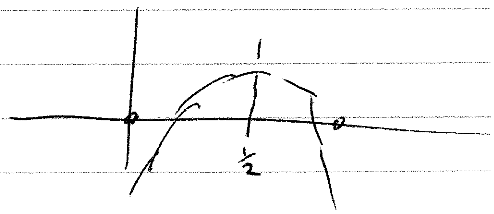

$$\mathrm{Im} \log\left[m^2 - x(1-x)p^2 - i\varepsilon\right] \overset{\varepsilon\downarrow 0}{=} \cancel{-\frac{\pi}{2}}\ \cancel{\frac{\pi}{2}} - \pi$$

$$\therefore \mathrm{Im}\, \Sigma(p^2) = \frac{h^2}{2(4\pi)^2} \int dx \left(-\frac{\pi}{\cancel{4}}\right)$$

We have to integrate over restricted region $x(1-x) > \frac{m^2}{p^2}$

Solve for region of integration. Boundaries are

$$-x^2 + x = \frac{m^2}{p^2}$$




$$x^2 - x + \frac{m^2}{p^2} = 0$$

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4m^2}{p^2}} \qquad \text{real if } p^2 \geq 4m^2$$

$$\mathrm{Im}\, \Sigma(p^2) = \frac{h^2}{2(4\pi)^2} \int_{x_-}^{x_+} dx \left(-\frac{\pi}{\cancel{4}}\right) \Theta(p^2 - 4m^2)$$

$$\operatorname{Im} \Sigma(p^2) = -\frac{\pi h^2}{2\cancel{4}(4\pi)^2}\sqrt{1-\frac{4m^2}{p^2}}\;\theta(p^2-4m^2)$$

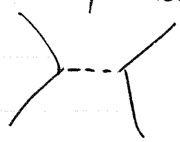
$$= \frac{-h^2}{32\cancel{64}\pi}\sqrt{1-\frac{4m^2}{p^2}}\;\theta(p^2-4m^2).$$

So propagator is (ignore wavefunction renorm.)

$$\frac{i}{p^2-M_x^2+iM_x\Gamma(p^2)} \qquad \Gamma(p^2) \equiv \frac{h^2}{32\cancel{64}\pi M}\sqrt{1-\frac{4m^2}{p^2}}\;\theta(p^2-4m^2)$$

Now if we scatter ~~say~~ $\varphi$ particles ~~the~~ with $E_{total} = M_x + \varepsilon$ for small $\varepsilon$, $\vec{p}_{tot} = 0$

Amplitude $\quad (ih)^2 \dfrac{i}{p^2-M_x^2+iM_x\Gamma(p^2)} = -i\mathcal{M}$



$$\mathcal{M} = \frac{h^2}{p^2-M_x^2+iM_x\Gamma(p^2)} \qquad p^2 = M_x^2 + 2M_x\varepsilon$$

$$\simeq \frac{h^2}{2M_x\varepsilon + iM_x\Gamma(M_x^2)}$$

Assumes a narrow resonance

$$\sigma \cancel{=} \propto \frac{1}{|2\varepsilon+i\Gamma|^2} = \frac{1}{4\varepsilon^2+\Gamma^2}$$

This is the Breit-Wigner line shape with width $\Gamma$, same as occurs in ordinary QM.

In Lecture 33, Mark gave a formula for the decay rate $\Gamma$: $A \to 1 + \cdots + n$

$$d\Gamma = \frac{1}{2m_A} |\mathcal{M}|^2 (2\pi)^4 \delta^4\left(p_A - \Sigma p_i\right) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

For $X \to \varphi\varphi$ this is

$$d\Gamma = \frac{1}{2M_X} h^2 (2\pi)^4 \left(p_X - p_1 - p_2\right) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

Can now show why this is.

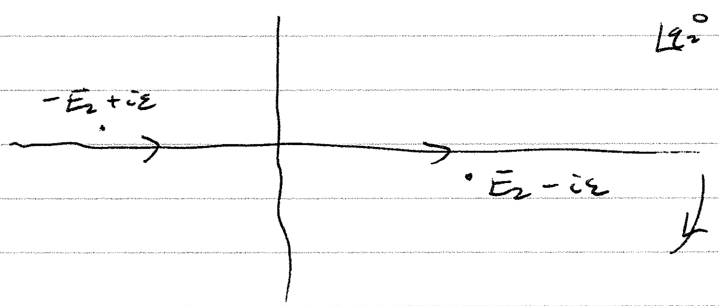
We had

$$\Sigma(p^2) = \frac{ih^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(p+q)^2 - m^2 + i\varepsilon} \frac{1}{q^2 - m^2 + i\varepsilon}$$

$$= \frac{ih^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 - m^2 + i\varepsilon} \frac{1}{q_2^2 - m^2 + i\varepsilon} (2\pi)^4 \delta^4(p - q_1 - q_2)$$

Do integral over $q_2^0$. Leave $\delta$ function alone. $\to$ it gives other poles, don't contribute to Im part though. Poles at

$q_2^0 = E_2 - i\varepsilon$, $q_2^0 = -E_2 + i\varepsilon$




Choose to close in LHP.

Residue $= \dfrac{1}{2E_2}$

$$\Sigma(p^2) = \frac{ih^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\frac{d^3q_2}{(2\pi)^4}\frac{1}{q_1^2 - m^2 + i\epsilon}\frac{i}{2E_2}(2\pi)^4\delta^4(p-q_1-q_2)$$

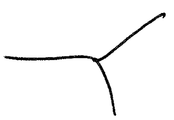
Repeat for $q_1^0$ integral

$$= -\frac{ih^2}{2}\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4\delta^4(p-q_1-q_2)$$

$$\therefore\ \Gamma = -\frac{1}{M_x}\,\mathrm{Im}\,\Sigma(M_x^2).$$

Can keep on doing integrals for more general case.

Note that we have found a relationship between a tree and a loop process.

$\Big|$  $\Big|^2 = \mathrm{Im}$ 

This is a special case of the optical theorem which will be the topic of the next lecture.

## Width & Lifetime

We found last time corrected propagator

$$\frac{i}{p^2 - M^2 + iM\Gamma(p)} = D_F(p)$$

Suppose for simplicity that $\Gamma$ is constant. Consider the Fourier transform of $D_F(p)$ to coordinate space

$$\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{i}{p^2 - M^2 + iM\Gamma}$$

Look at $p^0$ integral

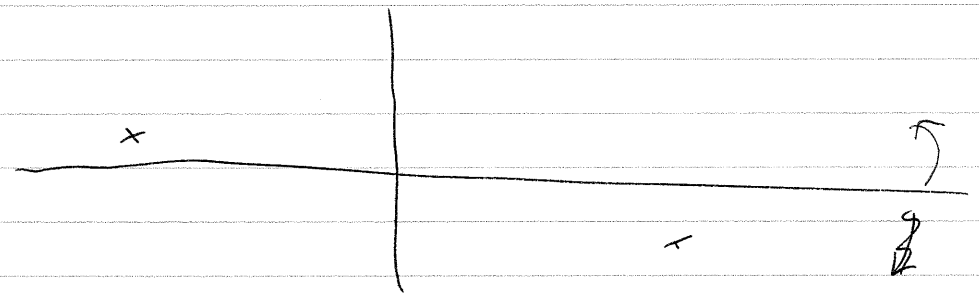
$$\int \frac{dp^0}{2\pi} e^{ip^0(x^0-y^0)} \frac{i}{(p^0)^2 - E_p^2 + iM\Gamma} \qquad E_p^2 = \vec{p}^{\,2} + M^2$$

Poles when $\quad (p^0)^2 = E_p^2 - iM\Gamma$

$$p^0 = \pm E_p \sqrt{1 - iM\frac{\Gamma}{E^2}} \qquad \Gamma \ll M$$

$$\simeq \pm E_p \left(1 - iM\frac{\Gamma}{2E^2}\right)$$

$$p^0 \simeq \begin{cases} E_p - iM\frac{\Gamma}{2}\frac{1}{E} \\ -E_p + iM\frac{\Gamma}{2}\frac{1}{E} \end{cases} \qquad E \simeq M$$



Close in LHP for $x^0 > y^0$

For $x^0 > y^0$ choose pole $p^0 \simeq -E_p + i\frac{\Gamma}{2}$

$$\int \frac{dp^0}{2\pi} e^{ip^0(x^0-y^0)} \frac{i}{(p^0)^2 - E_p^2 + iM\Gamma}$$

$$\simeq \int \frac{dp^0}{2\pi} e^{ip^0(x^0-y^0)} \frac{i}{(p_0 - E_p + i\frac{\Gamma}{2})(p_0 + E_p - i\frac{\Gamma}{2})}$$

$$= (2\pi i)\frac{i}{2\pi}\frac{1}{-2E_p + i\Gamma} e^{-iE_p(x^0-y^0)} e^{-\frac{\Gamma}{2}(x^0-y^0)}$$

$$= \frac{1}{2E_p - \underbrace{i\Gamma}_{\text{neglect}}} e^{-iE_p(x^0-y^0)} e^{-\frac{\Gamma}{2}(x^0-y^0)}$$

$\therefore$ Amplitude for disturbances to travel into future $\propto e^{-\Gamma/2\, t}$

$\Rightarrow$ Probability $e^{-\Gamma t} \Rightarrow \Gamma = \frac{1}{\tau}$.

---

Reminders: Lecture 18

$$d\sigma_{fi} = \frac{|\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - \cdots - k_n)}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{d^3k_1}{(2\pi)^3 2E_1} \cdots \frac{d^3k_n}{(2\pi)^3 2E_n}$$

$$\underbrace{{}_{out}\langle \vec{k}_1 \cdots \vec{k}_n | \vec{p}_1 \vec{p}_2 \rangle_{in}}_{S \text{ matrix.}} = \mathcal{M}_{fi} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - \cdots - k_n)$$

# The Optical Theorem

Recall : the S-matrix is the matrix whose elements are $\langle \text{out} | \text{in} \rangle$ for all possible in and out states.

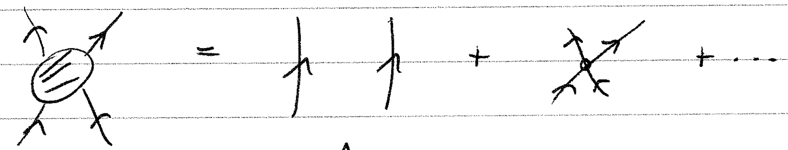
Then $S^{+} S = \sum {}_{in}\langle \psi | \psi' \rangle_{out} \, {}_{out}\langle \psi' | \psi \rangle_{in} = {}_{in}\langle \psi | \psi \rangle_{in}$

$= 1$

Similarly $S S^{+} = 1$.

The S-matrix is a unitary matrix. This is the sense in which probability is conserved in QFT.

Eg



straight through No interaction.

We write $S = 1 + iT$ , T transition matrix.

In terms of Feynman diagrams, T contains all the graphs with an interaction.

$$S^{+} S = 1 \Rightarrow (1 - i T^{+})(1 + iT) = 1$$

$$\therefore \; -i(T^{+} - T) + T^{+} T = 0$$

$$T^{+} T = -i(T - T^{+})$$

What does this mean?

Consider 2 particle states $|\vec{p}_1 \vec{p}_2\rangle$ and $|\vec{k}_1 \vec{k}_2\rangle$.
Then insert complete set of states $|\{\vec{q}_i\}\rangle$

$$\langle \vec{p}_1 \vec{p}_2 | T^\dagger T | \vec{k}_1 \vec{k}_2 \rangle =$$

$$= \sum_n \prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \langle \vec{p}_1 \vec{p}_2 | T^\dagger | \{\vec{q}_i\}\rangle \langle \{\vec{q}_i\} | T | \vec{k}_1 \vec{k}_2 \rangle$$

$$= \sum_n \prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \left( \langle \{\vec{q}_i\} | T | \vec{p}_1 \vec{p}_2 \rangle \right)^* \times$$

$$\times \langle \{\vec{q}_i\} | T | \vec{k}_1 \vec{k}_2 \rangle$$

$$= \sum_n \prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \mathcal{M}^*(p_1 p_2 \to \{q_i\}) \mathcal{M}(k_1 k_2 \to \{q_i\})$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - \Sigma q_i)(2\pi)^4 \delta^4(k_1 + k_2 - \Sigma q_i)$$

But $\delta^4(p_1 + p_2 - \Sigma q_i)\, \delta^4(k_1 + k_2 - \Sigma q_i)$

$$= \delta^4(k_1 + k_2 - \Sigma q_i)\, \delta^4(k_1 + k_2 - p_1 - p_2).$$

So

$$\langle \vec{p}_1 \vec{p}_2 | T^\dagger T | \vec{k}_1 \vec{k}_2 \rangle$$

$$= \sum_n \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \mathcal{M}^*(p_1 p_2 \to \{q_i\}) \mathcal{M}(k_1 k_2 \to \{q_i\})$$

$$(2\pi)^4 \delta^4(k_1 + k_2 - \Sigma q_i)\,(2\pi)^4 \delta^4(k_1 + k_2 - p_1 - p_2)$$

On the other hand

$$-i \langle \vec{p}_1 \vec{p}_2 | (T - T^\dagger) | \vec{k}_1 \vec{k}_2 \rangle$$

$$= -i \left( \mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2) \right) (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2)$$

$$\therefore\ -i \left( \mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2) \right)$$

$$= \sum_n \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \mathcal{M}^*(p_1 p_2 \to \{q_i\}) \mathcal{M}(k_1 k_2 \to \{q_i\})$$

$$\times (2\pi)^4 \delta^4(k_1 + k_2 - \Sigma q_i)$$

Suppose $k_1 = p_1$, $k_2 = p_2$. Then

$$-i \left( \mathcal{M}(p_1 p_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to p_1 p_2) \right)$$

$$= 2\, \mathrm{Im}\, \mathcal{M}(p_1 p_2 \to p_1 p_2)$$

Also RHS is

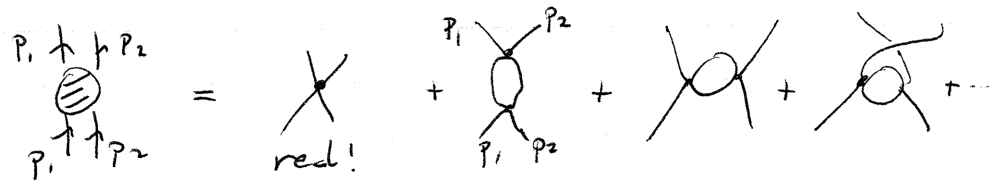
$$\sum_n \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \left| \mathcal{M}(p_1 p_2 \to \{q_i\}) \right|^2 (2\pi)^4 \delta^4(p_1+p_2-\Sigma q_i)$$

$$= 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}\ \sigma(p_1 p_2 \to \text{anything})$$

$$= 4 E_{cm} |\vec{p}_{cm}|\ \sigma_{tot}$$

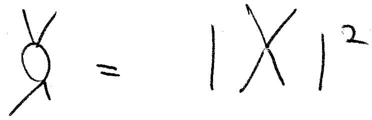
$$\therefore \boxed{\operatorname{Im} \mathcal{M}(p_1 p_2 \to p_1 p_2) = 2 E_{cm} |\vec{p}_{cm}|\ \sigma_{tot}}$$

Eg $\lambda \varphi^4$




Only one with im part

Diagrammatically Im



We specialized to 2 particle states. If we had used 1 particle states would get

$$2 \operatorname{Im} \mathcal{M}(p \to p) = \sum_n \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \left| \mathcal{M}(p \to \text{anything}) \right|^2$$

$$\times (2\pi)^4 \delta^4(p - \Sigma q_i)$$

$$\therefore \operatorname{Im} \mathcal{M}(p \to p) = M\, \Gamma_{total}$$

sign compared to yesterday: $\Sigma = -\mathcal{M}$.

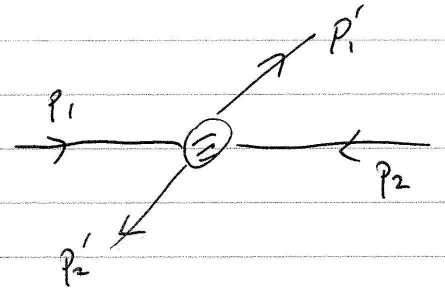
## Optical Theorem & Higgs Mass

~~Dar~~ Optical theorem constrains Higgs mass. Can't go into all the details, sketch idea.

Consider massless 2 particle scattering. Then optical theorem is

$$\mathrm{Im}\, \mathcal{M}(p_1 p_2 \to p_1 p_2) = s\, \sigma(p_1 p_2 \to \text{anything.})$$

For $2 \to 2$ scattering, can use Mandelstam variables. Work in CM frame.




$$p_1 = (E, \vec{p})$$
$$p_2 = (E, -\vec{p})$$
$$p_1' = (E, \vec{p}\,')$$
$$p_2' = (E, -\vec{p}\,')$$

$$s = (p_1 + p_2)^2 = (2E)^2 = E_{cm}^2$$

$$t = (p_1' - p_1) = -(2|\vec{p}|)^2 \cos\theta$$

$$= -(2E_{cm})^2 \cos\theta$$

$$s + t + u = 0$$

$\therefore$ Describe $\mathcal{M}$ in terms of $s, t$; equivalently, in terms of $s, \cos\theta$.

$$\mathcal{M}(s, \cos\theta) = 16\pi \sum_{\ell=0}^{\infty} a_\ell(s)\, P_\ell(\cos\theta)\,(2\ell+1)$$

LHS of optical theorem is

$$16\pi \sum_{\ell=0}^{\infty} \operatorname{Im} a_\ell(s)\, P_\ell(1)\,(2\ell+1)$$

$$= 16\pi \sum_{\ell=0}^{\infty} \operatorname{Im} a_\ell(s)\,(2\ell+1)$$

Now look at RHS. Suppose CM energy is too low to create new particles (toy model.) Then only process contributing to $\sigma$ is $p_1 p_2 \to p_1' p_2'$, with amplitude $\mathcal{M}(s, t)$.

$$\sigma = \frac{1}{64\pi^2 s} \int d\Omega\, |\mathcal{M}(s, t)|^2$$

$$= \frac{1}{64\pi^2 s}\, 2\pi \int_{-1}^{1} d(\cos\theta)\, (16\pi)^2 \sum_{\ell,\ell'=0}^{\infty} a_\ell(s)\, a_{\ell'}^{*}(s)$$

$$(2\ell+1)(2\ell'+1)\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)$$

$$\cancel{= \frac{8\pi}{s} \int_{-1}^{1} d(\cos\theta)}$$

$$= \frac{8\pi}{s} \sum_{\ell,\ell'=0}^{\infty} a_\ell(s)\, a_{\ell'}^{*}(s)\,(2\ell+1)(2\ell'+1) \int_{-1}^{1} d(\cos\theta)\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)$$

Orthogonality of $P_\ell$: $\displaystyle\int_{-1}^{1} dx\, P_\ell(x)\, P_{\ell'}(x) = \frac{2\,\delta_{\ell\ell'}}{2\ell+1}$

$$\sigma = \frac{16\pi}{s} \sum_{l=0}^{\infty} |a_l(s)|^2 (2l+1)$$

$\therefore$ RHS of optical thm:

$$16\pi \sum_{l=0}^{\infty} |a_l(s)|^2 (2l+1)$$

Optical theorem requires

$$\sum_{l=0}^{\infty} (2l+1) \left[ |a_l(s)|^2 - \mathrm{Im}\, a_l(s) \right] = 0$$

A more sophisticated version of this yields

$$|a_l(s)|^2 = \mathrm{Im}\, a_l(s) \qquad \text{for small } s.$$

$$|a_l(s)|^2 \leq \mathrm{Im}\, a_l(s) \qquad \text{more generally.}$$

$$a_l(s) = x + iy$$

$$x^2 + y^2 \leq y$$

$$x^2 \leq y(1-y)$$

Also $a_l(s) = re^{i\theta}$

$$r^2 \leq r\sin\theta \quad \Rightarrow \quad r < 1 \quad \Rightarrow \quad |y| < 1$$

$$\therefore \quad x^2 \leq \frac{1}{4}, \quad |\mathrm{Re}\, a_l(s)| \leq \frac{1}{2}.$$

Turns out that amplitude for longitudinal WW scattering in $l=0$ is $a_0 = -\dfrac{G_F m_H^2}{4\pi\sqrt{2}} \Rightarrow m_H \lesssim 850\,\text{GeV}.$