Nonperturbative effects in Matrix Models and Topological Strings

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collaboration with M.Mariño, R. Schiappa
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Introduction and Motivation
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Instantons & Large Order: The Anharmonic Oscillator

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**Introduction**

**Topological strings**

Consider the \textbf{A-model} on a Calabi-Yau $X$

$$F(Q, g_s) = \sum_{d,g} N_{d,g} Q^d g_s^{2g-2}, \quad Q = e^{-t}$$

- count worldsheet instantons
- perturbative in $Q, g_s$

\[\downarrow\text{mirror symmetry}\downarrow\]

\textbf{B-model} on $X_{\text{mirror}} \rightarrow \text{compute } F_g(Q) \text{ exactly in } Q$

...but can we go beyond perturbation theory in $g_s$?
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Non-perturbative and Large Order

Why going non-perturbative?

- A better understanding of (topological) strings
  - instanton effects → dynamics?
  - new topological invariants?
- Compute *perturbative* amplitudes using non-perturbative methods?
  - WKB-like tools?

Large Order behavior & Nonperturbative effects

- QM, QFT: Standard relation between instanton effects and large-order behavior of the perturbation series
- Asymptotics of $\frac{1}{N}$-expansion of gauge theories controlled by nonperturbative corrections $\sim e^{-N}$
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- D-brane instanton effects in string dual

[Alexandrov Kazakov Kutasov]
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Applications to Topological String Theory

If the gauge theory has a string dual:

- Instanton effect in gauge theory $\leftrightarrow$ asymptotics of string amplitudes

  - Natural non-perturbative completion
  - Can be tested with asymptotics of string amplitudes!
  - Information about analytic structure of topological string free energy
  - Nontrivial check of conjectural dualities
  - New conjectures about asymptotics of enumerative invariants

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- Matrix models in double-scaling limit $\leftrightarrow$ noncritical string theory
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Matrix Models and Topological Strings

B-model on some local CYs $\overset{\text{large } N \text{ dual}}{\longleftrightarrow}$ Matrix model

[Dijkgraaf Vafa]

This also works for mirrors of toric geometries!

[Mariño; Bouchard Klemm Mariño Pasquetti]

new formalism to compute open & closed B-model amplitudes:

Topological string amplitudes $F_g$ behave like matrix model correlators

Recursive, geometric reformulation of matrix model $1/N$-expansion: all information encoded in spectral curve

[Eynard Orantin]

- Spectral curve for TS on mirror of toric CY: mirror curve
  $$\Sigma_t(u,v) = w^+ w^-$$

- recursive matrix model formalism $\rightarrow$ generate TS amplitudes

- no holomorphic ambiguity

- at large radius: mirror to topological vertex, but valid anywhere in moduli space
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Instanton effects and Large Order behavior

A Quantum mechanics example

Consider the anharmonic oscillator with Hamiltonian

\[ H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda x^4}{4}. \]

Take the perturbative expansion of the ground-state energy,

\[ S_E(\lambda) = \sum_k E_k \lambda^k. \]

- \( S_E \) is in principle expected to have zero radius of convergence, \( R = 0! \)
  [Dyson]

- Indeed here: \( R > 0 \) would imply that the perturbative series describes the physics also for \( \lambda < 0 \), where the state is unstable and the particle escapes.
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\[ E(\lambda) \]
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\[ \text{Re}(E(\lambda)) \]

\[ \text{Im}(E(\lambda)) \]

\[ \text{lifetime} \]
The anharmonic oscillator

Let $S_E(\lambda) = \sum_{k \geq 0} E_k \lambda^k$ be the formal, divergent expansion of the ground state energy $E(\lambda)$.

- $E(\lambda)$ is an analytic function of the coupling $\lambda$ in the cut complex plane.
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\[
E(\lambda) = \frac{1}{2\pi i} \oint d\lambda' \frac{E(\lambda')}{\lambda' - \lambda}
\]

→ We can deform the Cauchy representation to the dispersion relation

\[
E_k = \frac{1}{2\pi i} \int_{-\infty}^{0} d\lambda' \frac{\text{Disc}(E(\lambda'))}{\lambda'^{k+1}}
\]
\[ E_k = \frac{1}{2\pi i} \int_{-\infty}^{0} d\lambda \frac{\text{Disc}(E(\lambda))}{\lambda^{k+1}} \]

- This result is rigorous and exact
- The perturbation coefficients are related to the lifetime of the state in the unstable potential with negative coupling ↔ instanton effect at \( \lambda < 0 \)

\[ \text{Disc}(E(\lambda)) = ? \]

Consider \( I(\lambda) = \int_{-\infty}^{\infty} e^{-(x^2 + \lambda x^4)} dx \): Analytic continuation to \( \lambda < 0 \).
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![Diagram of complex plane with contours C_+ and C_- and saddle points S_1 and S_2]

\[ E_k \sim \frac{\mu_1}{2\pi} \mathcal{A}_{\text{inst}}^{-k-b} \Gamma(k + b)(1 + \frac{\mathcal{A}_{\text{inst}}}{(b+k-1)} \mu_2 + O(\frac{1}{k^2})) \rightarrow \]

anharmonic oscillator: \( E_k \sim (-1)^{k+1} \frac{\sqrt{6}}{\pi^2} 3^k \Gamma(k + \frac{1}{2}) \) [Bender Wu]
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\[ \uparrow 1\text{-loop} \]

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\[ \text{saddle-point expansion} \]

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\[ \mathcal{A}_{\text{inst}} = 2 \int_{0}^{x_0} \sqrt{2V(x)} dx = -\frac{1}{3} \rightarrow \text{action of tunneling-instanton} \]

\[ E_k \sim \frac{\mu_1}{2\pi} \mathcal{A}_{\text{inst}}^{k-b} \Gamma(k+b)(1 + \frac{\mathcal{A}_{\text{inst}}}{b+k-1})\mu_2 + O\left(\frac{1}{k^3}\right) \rightarrow \text{anharmonic oscillator: } E_k \sim (-1)^{k+1} \frac{\sqrt{6}}{3} 3^{k/2} (k - \frac{1}{2}) \]
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Analogously, \( \text{Disc}(E(\lambda)) = \frac{Z_{1-\text{inst}}}{Z_{0-\text{inst}}} \)

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Matrix models in $1/N$ expansion

- **Partition function**
  
  $$Z = \frac{1}{\text{vol}(U(N))} \int dM e^{-\frac{1}{g_s} \text{Tr} V(M)} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dz_i}{2\pi} e^{N^2 V_{\text{eff}}(z_i)}$$

- **Effective potential**
  
  $$V_{\text{eff}}(z_i) = V(z_i) - 2 \frac{t}{N} \sum_{i \neq j} \log |z_i - z_j| \rightarrow \text{Coulomb repulsion} \rightarrow \text{eigenvalues spread out over interval } C$$

- The object we are interested in is the free energy;

  $$F(t) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

  where $t = g_s N$ is the 't Hooft parameter

- $t$ fixed: expansion in $g_s \leftrightarrow$ expansion in $\frac{1}{N}$
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- Partition function

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- Effective potential $V_{\text{eff}}(z_i) = V(z_i) - 2 \frac{t}{N} \sum_{i \neq j} \log |z_i - z_j| \to$ Coulomb repulsion $\to$ eigenvalues spread out over interval $C$

- The object we are interested in is the free energy:

$$F(t) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

where $t = g_s N$ is the 't Hooft parameter

- $t$ fixed: expansion in $g_s \leftrightarrow$ expansion in $1/N$
Matrix models in $1/N$ expansion

- **Partition function**

\[
Z = \frac{1}{\text{vol}(U(N))} \int dMe^{-\frac{1}{gs} \text{Tr}V(M)} = \frac{1}{N!} \int \prod_{i=1}^N \frac{dz_i}{2\pi} e^{N^2 V_{\text{eff}}(z_i)}
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1-cut 2-cut
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Here: Consider 1-cut case only
The planar solution

- When $N \to \infty$, the distribution of eigenvalues becomes continuous and one can write

$$V_{\text{eff}}(z) = V(z) - \frac{1}{2\pi} \int (y(z + i0) - y(z - i0)) \log |z - z'| dz,$$

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- The effective potential is constant along the cut and has a saddle point at $x_0$:

- **Instanton configuration**: an eigenvalue from the endpoint of the cut moves to the saddle of the effective potential barrier
The instanton action is

\[ \mathcal{A}_{\text{inst}} = N \int_{a}^{x_0} y(z) dz \]

[David; Shenker]
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\[ \mathcal{A}_{\text{inst}} = N \int_{a}^{x_0} y(z) \, dz \]

[David; Shenker]

Geometrically, \( \mathcal{A}_{\text{inst}} \) is a contour integral from endpoint of the cut to singularity of spectral curve

[Seiberg Shih]
Instanton analysis

We expect a relation instantons ↔ large-order analogous to the anharmonic oscillator:

\[ F_g = \frac{1}{2\pi} \int_0^\infty ds \frac{\text{Disc}(F(\sqrt{s}))}{s^{g+1}} = \mu_1 \frac{\mathcal{A}_{\text{inst}}^{b-2g}}{\pi} \Gamma(2g + b) \left( 1 + \frac{\mathcal{A}_{\text{inst}}}{2g + b - 1} \mu_2 + O\left(\frac{1}{g^2}\right) \right) \]

- The large-order behavior is controlled by Disc\((F(g_s))\)
- The discontinuity of \(F(g_s)\) is again given by

\[ \text{Disc}(F(g_s)) = \frac{Z_N^{(1-\text{inst})}(g_s)}{Z_N^{(0-\text{inst})}(g_s)} \]

- \(Z_N^{(1-\text{inst})}\) corresponds to one eigenvalue passing through the nontrivial saddle \(x_0\) of the spectral curve
- \(Z_N^{(1-\text{inst})}\) factorizes as

\[ Z_N^{(1)} = Z_N^{(0)} \int_{C_{x_0}} dz \langle \text{det}(z - M)^2 \rangle_{N-1}^{(0)} \exp\left(-\frac{V(z)}{g_s}\right) \]
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$\uparrow$ \hspace{2cm} $\uparrow$

1-loop, leading \hspace{2cm} 2-loop, subleading

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• $W_{g,h}$ determined recursively from spectral curve by matrix model loop equations

[Ambjørn Chekhov Kristjansen Makeenko; Eynard Orantin]

• The remaining ingredient is

$$\frac{Z^{(0)}_{N-1}}{Z^{(0)}_N} = \exp \left( \sum_{g=0}^{\infty} g_s^{2g-2} (F_g(g_s(N-1)) - F_g(g_sN)) \right),$$

and we find in saddle-point analysis

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Disc\( (F) \) = \( \mu_1 g_s^{1/2} \exp\left(-\frac{\mathcal{A}_{\text{inst}}}{g_s}\right)(1 + g_s \mu_2 + \cdots) \)

Explicitly:

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\mu_1 = \frac{(a - b)}{4} \sqrt{\frac{1}{2\pi y'(x_0)((x_0 - a)(x_0 - b))^{3/2}}} e^{-\frac{1}{g_s} \mathcal{A}_{\text{inst}}}
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- Disc\( (F) \) depends only on the spectral curve of the matrix model, not on the potential
- ↓ B-model formalism
- unambiguously defined for topological strings on mirrors of toric geometries
- \( a, b, x_0 \) depend on \( 't \) Hooft parameter \( t \)
- Disc\( (F) \) \( \sim e^{-N\mathcal{A}_{\text{inst}}/t} \) → non-perturbative
- \( \mu_1 \) has been computed before, but the result is not valid off criticality
  
  [Hanada Hayakawa Ishibashi Kawai Kuroki Matsuo Tada]
- We have computed Disc\( (F) \) to two loops → \( \mu_1, \mu_2 \)
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String interpretation of the instanton effects

Instanton action in the double-scaling limit of matrix model

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\[ = W(x_0) - W(a) \]

\[ \leftrightarrow \]

disk amplitude for D-instanton in noncritical string theory \( \rightarrow \) ZZ-brane

[Alexandrov Kazakov Kutasov]

\[ \downarrow \]
difference between disk amplitudes of FZZT branes
\[ W_{\text{FZZT}}(a) - W_{\text{FZZT}}(x_0) \]

Is there a similar story for topological string theory?

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two branes located at \( a, x_0 \) with difference between superpotentials \( W(x_0) - W(a) \)
\[ \rightarrow \] define domain wall in underlying type II theory, with tension given by \( \mathcal{A}_{\text{inst}} \)

Unlike the B-branes, this domain wall can couple to the complex structure!
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Examples

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![Diagram]

- Quartic Matrix Model
  - double-scaling limit
  - 2d Gravity
  - Local Curve $X_p$ (double-scaling limit)
  - Hurwitz Theory ($p \to \infty$)

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Numerical analysis: Richardson transformation

$F_g$ are only available to limited genus, how to extract the asymptotics as $g \to \infty$? → Richardson transformation.

Given a sequence $(S_g)$,

$$S_g = s_0 + \frac{s_1}{g} + \frac{s_2}{g^2} + \cdots,$$

the subleading corrections up to order $\frac{1}{g^n}$ can be removed defining

$$A(g, n) = \sum_{k=0}^{N} S_{g+k}(g+k)^n (-1)^{g+n} \frac{1}{k!(n-k)!}.$$

If $S_g$ truncates at $1/g^n$, this gives exactly $s_0$: for $n=1$;

$$S_g = s_0 + \frac{s_1}{g} \to A(g, 1) = -(s_0 + \frac{s_1}{g}) + (s_0 + \frac{s_1}{g+1})(g+1) = s_0$$

$A(g, n) = s_0 + O\left(\frac{1}{g^{n+1}}\right) \to$ accelerates convergence
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The quartic matrix model

Consider the matrix model with quartic potential

\[ V(M) = \frac{1}{2} M^2 + \lambda M^4 \]

- spectral curve:

\[ y(z) = (1 + 8\lambda a^2 + 4\lambda z^2) \sqrt{z^2 - 4a^2} \]

\[ \pm 2a = \text{endpoints of the cut,} \]

\[ a(\lambda) = \frac{1}{24\lambda} \left( -1 + \sqrt{1 + 48\lambda} \right) \]

[Brézin Itzykson Parisi Zuber]

- Critical point at \( \lambda = -\frac{1}{48} \)

- The free energy in \( \frac{1}{N} \)-expansion can be computed by standard methods

[Bessis Itzykson Zuber]

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19/28
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- The quartic matrix model
- 2d gravity
- The local curve
- Hurwitz Theory

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The numerical asymptotics for the instanton action, along with the matrix prediction, at $\lambda = -0.1$

The leading asymptotics for $F_{g}^{\text{quart}}(\lambda)$, divided by the one-loop matrix prediction

The subleading asymptotics, divided by the two-loop prediction
2d gravity

- Taking $N \to \infty$ in a standard matrix model retains only planar surfaces unless one simultaneously takes $\lambda \to \lambda_c$ where higher-genus contributions are enhanced as $F_g \propto (\lambda - \lambda_c)^{(2-\gamma)(1-g)}$: double-scaling limit $\to$ 2d gravity
  [Gross Migdal; Douglas Shenker]

- limit discretized surface $\to$ continuum

- The perturbative amplitudes are governed by the Painlevé I equation fulfilled by the specific heat $u(z) = F''(z)$,
  \[
  u^2 - \frac{u''}{6} = z
  \]

- can compute $F_g$ to arbitrary genus

- The instanton action and 1-loop factor are
  \[
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  \]
  [David]
2d gravity

- Taking $N \to \infty$ in a standard matrix model retains only planar surfaces unless one simultaneously takes $\lambda \to \lambda_c$ where higher-genus contributions are enhanced as $F_g \propto (\lambda - \lambda_c)^{(2-\gamma)(1-g)}$: double-scaling limit $\to$ 2d gravity

  [Gross Migdal; Douglas Shenker]

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The leading asymptotics, divided by the one-loop prediction

The subleading asymptotics, divided by the two-loop prediction
The local curve

Consider A-model topological strings on the local curve

\[ X_p = O(p) \oplus O(2 - p) \to \mathbb{P}^1, \quad p \in \mathbb{Z}. \]

- This is a toric Calabi-Yau threefold with one Kähler modulus
- The potential is unstable for all \( p > 2 \)
- The free energy can be computed using the topological vertex or local Gromov-Witten theory
  [Aganagic Klemm Mariño Vafa; Bryan Pandharipande]
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The spectral curve corresponding to the matrix description of the mirror B-model is

\[ y(z) = \frac{2}{z} \left( \tanh^{-1} \left( \frac{\sqrt{(z-a)(z-b)}}{z - \frac{a+b}{2}} \right) - p \tanh^{-1} \left( \frac{\sqrt{(z-a)(z-b)}}{z + \sqrt{ab}} \right) \right), \]

[Mariño]

- The endpoints of the cut \( a, b \) depend on the exponential of the Kähler parameter \( Q \) via the mirror map:

\[
\begin{align*}
a &= \frac{(1 + \sqrt{\zeta})^2}{(1 - \zeta)^p}; & b &= \frac{(1 - \sqrt{\zeta})^2}{(1 - \zeta)^p} \\
Q &= (1 - \zeta)^p (p-2) \zeta
\end{align*}
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- The B-model matrix formalism provides a nonperturbative completion that is testable with the large-order behaviour of the perturbative amplitudes \( F_g(Q) \).
- Using the topological vertex, we computed \( F_g \) up to genus 9 (genus 7) for \( p=3 \) (\( p=4 \)).
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The numerical asymptotics for the instanton action, along with the matrix prediction, at $\zeta = .15, p = 3$

The leading asymptotics for $F_{g}^{p=3}$, divided by the one-loop prediction

The subleading asymptotics, divided by the two-loop prediction
Hurwitz Theory

- Hurwitz theory counts branched covers of Riemann surfaces obtained as a special limit of the local curve $X_p$:

$$p \to \infty, \ g_s \to 0, \ Q \to 0; \ g^H = Npg_s, \ Q_H = \frac{(-1)^p}{(g_s N)^2} Q$$

$$F^H = \sum_{g \geq 0} N^{2-2g} \sum_{d \geq 0} Q_H^{d} H_{g,d}^{\mathbb{P}^1} (1^d) \cdot \frac{g^{2g-2+2d}}{(2g-2+2d)!}$$

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$$\chi e^{-\chi} = Q^H, \ a_H(\chi) = (1 + \sqrt{\chi})^2, \ b_H(\chi) = (1 - \sqrt{\chi})^2$$

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Conclusion and Outlook

- We have computed nonperturbative effects for a generic matrix model.
- The B-model formalism defines a nonperturbative completion for topological strings on local geometries.
- All can be tested with the large-order behavior of the string perturbation series: agreement to very high precision.

Challenges ahead:
- multi-cut case
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