

Encoding of (part of) $N=4$ superconformal algebra into 4 twisted differential operators

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October 28th, 2009

CALTECH

DEFINING THE TWIST FOR THE N=4 THEORY

We begin with the N=4 action

$$S = \frac{1}{g^2} \int d^4x \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - i \bar{\lambda}_A^{\dot{\alpha}} \mathcal{D}_{\dot{\alpha}\beta} \lambda^{\beta A} - i \lambda_\alpha^A \mathcal{D}^{\alpha\dot{\beta}} \bar{\lambda}_{A\dot{\beta}} + \frac{1}{2} (D_\mu \bar{\phi}_{AB}) (D^\mu \phi^{AB}) \right. \\ \left. - \sqrt{2} \bar{\phi}_{AB} \{ \lambda^{\alpha A}, \lambda_\alpha^B \} - \sqrt{2} \phi^{AB} \{ \bar{\lambda}_A^{\dot{\alpha}}, \bar{\lambda}_{\dot{\alpha}B} \} + \frac{1}{8} [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right\}$$

where

$$\delta_\epsilon A^\mu = \bar{\epsilon} \gamma^\mu \lambda, \quad \delta_\epsilon \phi^{AB} = \epsilon^A \lambda^B$$

The N=4 theory can be obtained by dimensionally reducing the N=1 d=10 theory. In fact, for maximal supersymmetry in dimension d with sixteen fermionic degrees of freedom, the spinors are organized as:

$$\left\{ \begin{array}{lll} N = 1 & d = 10 & \epsilon^\alpha = 16 \\ N = 2 & d = 8 & \epsilon^\alpha, \epsilon_{\dot{\alpha}} = 8 + 8 \\ N = 4 & d = 4 & \epsilon^{a\alpha}, \epsilon_{a\dot{\alpha}} = 4(2 + 2) \end{array} \right.$$

It is well known that the 16 supersymmetries as a whole close only up to field equations. In a light-cone superspace approach, one can manifest 8 off-shell susy, however these eight supersymmetries are selected in a non-lorentz covariant way. Here we will show that by doing a twist, one can covariantly select **9 susy generators that closes “off-shell”**, namely under these transformations

$$\{\delta, \hat{\delta}\} \sim -2i\bar{\epsilon}\gamma^\mu\hat{\epsilon}\partial_\mu - 2i\delta^{gauge}(\bar{\epsilon}\gamma^\mu A_\mu\hat{\epsilon})$$

Then we will see how it extends to the case of superconformal susy, with its 32 generators.

Although it is not the subject of this talk, let us mention that **the gauge transformation in the right hand side of the closure relations can be eliminated**, provided one introduces new fields called shadow fields. Such fields also make possible the definition of the action of supersymmetry on the Faddeev Popov ghosts of the gauge symmetry.

Let us come to the definition of the twist.

In Euclidean space the Lorentz group is $SO(4)=SU(2)\otimes SU(2)$ and the R-symmetry group $SO(6)$. The twist is done by taking the diagonal of one of the $SU(2)$ subgroup of the R-symmetry group $SO(6)$ and one of the $SU(2)$ of the Lorentz group. Thus we have the new Lorentz group $SO'(4)$ along with the remaining global $SL(2, R)$ group. That is

$$\begin{aligned} \text{lowering symmetry twist operation} : \quad & SO(4) \otimes SO(6) = SU(2)_L \otimes SU(2)_R \otimes SO(6) \\ & \rightarrow SO(4)' \otimes SL(2, R) \end{aligned}$$

Under the new Lorentz group the susy generators become scalars, vectors and (anti)self-dual tensors. The 16 supercharges are decomposed as follows

$$\text{generator twist operation} : \quad (Q_\alpha^a, \bar{Q}_{a\dot{\alpha}}) \rightarrow (\mathbf{Q}_0, \bar{\mathbf{Q}}_0, \mathbf{Q}_\mu, \bar{\mathbf{Q}}_\mu, \mathbf{Q}_{\mu\nu^-}, \bar{\mathbf{Q}}_{\mu\nu^-})$$

We have put in bold the 9 charges that turn out to build an off-shell closed algebra [1].

The fermions are twisted as follows

$$\textit{fermion twist operation} : (\lambda_\alpha^a, \lambda_{a\dot{\alpha}}) \rightarrow (\eta, \bar{\eta}, \psi_\mu, \bar{\psi}_\mu, \chi_{\mu\nu^-}, \bar{\chi}_{\mu\nu^-}) \equiv (\eta^\alpha, \psi^\alpha, \chi^{I\alpha})$$

where $\alpha = 1, 2$ labels the barred and unbarred. So the original $16 = 4 \times 4$ becomes $16 = (2 + 2 \times 4 + 2 \times 3)$ after the twist. Antiselfdual 2-forms have been relabelled as $SU(2) \subset SO'(4)$, ($I=1,2,3$), using the 3 Kahler 2-forms $J_{\mu\nu^-}^I$, with

$$J_{\mu\nu^-}^I X^{\mu\nu^-} = X^I$$

The 6 components of the $SO(6)$ -valued scalar are twisted as follows

$$\textit{scalar twist operation} : \phi^{ab} \rightarrow (\Phi, L, \bar{\Phi}, h_{\mu\nu^-}) \equiv (\phi^i, h^I)$$

The symmetry with 9 generators can be now closed off-shell, by introducing an auxiliary fields with 7 components. In this first twist, it is made of a vector and selfdual -form, $T_\mu, H_{\mu\nu^-}$.

So we must find a way to build the action of the 9 generators on the following representation

$$A_\mu, \Phi, \bar{\Phi}, L, h_{\mu\nu^-} \quad \Psi_\mu, \bar{\Psi}_\mu, \chi_{\mu\nu^-}, \bar{\chi}_{\mu\nu^-}, \eta, \bar{\eta} \quad T_\mu, H_{\mu\nu^-}$$

Some conventional explanation

In the language of supersymmetry, one can understand what is going on by coming back to 10 dimensions. For the 10 dimensional SYM action, a set of 7 non interacting scalar fields G_a ($a = 1, \dots, 7$), which count for the 7 missing bosonic degrees of freedom, was proposed in [2], with

$$\begin{aligned}\delta A_\mu &= i\bar{\epsilon}\hat{\Gamma}_\mu\Psi \\ \delta\Psi &= \hat{\Gamma}^{\mu\nu}F_{\mu\nu}\epsilon + 4G_av_a \\ \delta G_a &= -\frac{i}{4}\bar{v}_a\hat{\Gamma}^\mu D_\mu\Psi\end{aligned}$$

The corresponding susy transformations depend on the ordinary Majorana–Weyl parameter ϵ and on seven other spinor parameters v_a . The commuting spinor parameters ϵ and v_a must be constrained to have off-shell closure and it was found that the a solution can only be covariant under $SO(1, 1) \times Spin(7) \subset SO(1, 9)$. The constraints of off-shell closure between the parameters was shown to impose that one cannot have more than 9 off-shell susy transformations, but, no solution could be written in terms of covariant entities.

All this becomes very simple by twist [1], using the intuition from TQFT. In 8 dimensions, the 16 generators can be decomposed on irreducible representations after twist, as follows

$$(Q^\alpha, Q_{\dot{\alpha}}) \rightarrow (\mathbf{Q}_0, \mathbf{Q}_\mu, Q_{\mu\nu^-})$$

Then the $9=1+8$ generators Q_0 and Q_μ can be computed within the methods of TQFT as satisfying the algebra

$$Q_0^2 = Q_\mu Q_\nu + Q_\nu Q_\mu = 0, \quad Q_0 Q_\mu + Q_\mu Q_0 = \partial_\mu$$

which means off-shell closure, modulo gauge transformations. The non closure relations have been cornered in the sector of the selfdual generator $Q_{\mu\nu^-}$. (These properties survive after dimensional reduction.) In $D=8$, the auxiliary field is a self-dual 2-form $T_{\mu\nu^-}$, which express the 7 components of the 7 auxiliary fields G_a , in a $Spin(7) \subset Spin(8)$ invariant decomposition.

An off-shell closed representation of the $N = 2, d = 8$ theory is thus

$$A_\mu, \Phi, \bar{\Phi} \quad \Psi_\mu, \chi_{\mu\nu^-}, \eta \quad T_{\mu\nu^-}$$

In curved space this formulation requires the existence of a constant spinor, which allows one to map all spinors on forms, using triality. In fact the existence of such a constant spinor warrants the existence of a $Spin(7) \subset Spin(8)$ invariant tensor, which allows one to split any given 2-form in a selfdual and antiselfdual 2-form, according to the $Spin(7) \subset Spin(8)$ invariant decomposition $28 = 21 \oplus 7$. So, the 8-dimensional twist only preserves the $Spin(7) \subset Spin(8)$ invariance. Then one fermion and its auxiliary form belong the selfdual representation of dimension 7. An $SU(4)$ invariant decomposition with off-shell closure can be also done, using holomorphic representations of the fields,

One finally arrives by dimensional reduction in 4 dimensions, where the 7 auxiliary field decompose into a vector and a self-dual tensor ($T_\mu, H^I \sim H_{\mu\nu^-}$). The **$\mathbf{9} = \mathbf{1} + \mathbf{1} + \mathbf{4} + \mathbf{3}$ off-shell susy generators** in four dimensions are covariantly expressed in the twisted theory as

$$(\mathbf{Q}_0, \bar{\mathbf{Q}}_0, \mathbf{Q}_\mu, \mathbf{Q}_{\mu\nu^-})$$

One can show that the twist operation is stable under quantum corrections.

The 9 parameters of the theory for maximal supersymmetry with off-shell closure are thus

$$D = 8 \quad (\epsilon, \epsilon_\mu)$$

$$D = 7 \quad (\epsilon, \bar{\epsilon}, \epsilon_\mu)$$

$$D = 4 \quad (\epsilon, \bar{\epsilon}, \epsilon_\mu, \epsilon_{\mu\nu})$$

About the third twist

The passage by twist from the superPoincare representation to the firstly twisted representation is a linear mapping between fields, using σ matrices, giving equations that are invariant under a subgroup of the product of Lorentz and R symmetries, eg $SO'(4) \times SL(2, R) \subset SO(4) \times SO(6)$.

However, there is another twisted version in which the N=4 theory can be understood as a sort of complexified N=2 theory. This thirdly twisted theory can be obtained by another mapping from the firstly twisted theory, the main difference being

$$\chi_{\mu\nu^-} \rightarrow \chi_{\mu\nu^+}, \quad \phi^{ab} \sim (\Phi_i, h^I) \rightarrow (V_\mu, \Phi, \bar{\Phi})$$

Then the fields can be organized into a twisted N=2 complexified SYM theory with the field content

$$A_\mu + iV_\mu, \quad \Phi + i\bar{\Phi}, \dots$$

and the transformation laws can be obtained by complexification of those of the $N = 2$ theory. There are interesting relationships with the complex Chern Simons theory.

MORE ON THE N=4 ACTION AND ITS SYMMETRIES

We will elaborate on the existence of an **SL(2,R) covariant formulation**. To construct the N=4 action, one can use 6 of its twisted susy generators as follows [1]

$$s^\alpha = \begin{pmatrix} Q_0 \\ \bar{Q}_0 \end{pmatrix} ; \quad \delta^\alpha = \begin{pmatrix} k^\mu Q_\mu \\ k^\mu \bar{Q}_\mu \end{pmatrix}$$

These 4 odd graded operators satisfy a **nilpotent differential algebra**

$$\{s^\alpha, s^\beta\} = \sigma^{i\alpha\beta} \delta_{gauge}(\Phi_i) , \quad \{\delta^\alpha, \delta^\beta\} = \sigma^{i\alpha\beta} \delta_{gauge}(|k|^2 \Phi_i) , \quad \{s^\alpha, \delta^\beta\} = \epsilon^{\alpha\beta} (\mathcal{L}_k + \delta_{gauge}(i_k A))$$

The coefficients $\sigma^{i\alpha\beta}$ are for 2X2 $SL(2, R)$ matrices. $\mathcal{L}_\kappa = i_\kappa d + di_\xi$. Explicit computation shows the existence of the 4 graded scalar equivariant differential operators $s_{(c)}^\alpha$ and $\delta_{(c)}^\alpha$. They have **equivariant closure relations which donnot involve equations of motion**. In this sense they constitute an interesting subalgebra of the N=4 twisted conformal supersymmetry.

The geometrical meaning of the differential operators s^α and δ^α is shown by TQFT equations like :

$$(s + \delta + \bar{s} + \bar{\delta})(A + c) + (A + c)^2 = F + \psi + \bar{\psi} + g(k)(\eta + \bar{\eta}) + g(J^I k)(\chi^I + \bar{\chi}^I) + (1 + |k|^2)(\bar{\Phi} + L + \Phi)$$

that can be used to determine the action of s and δ , independently of the notion of Poincare supersymmetry.

Remember that the indices $\alpha, \beta, \gamma = 1, 2$ $i, j = 1, 2, 3$ are for to the fundamental and adjoint representations of an $SL(2, R)$ symmetry,,which is what is left from the $R = SO(6)$ symmetry, in the first twist. Capital indices for the fields $I = 1, 2, 3$ are devoted to the adjoint representation of the $SU(2)$ symmetry.

The fields, including the 7 auxiliary fields, are arranged as multiplets of $SL(2, R) \times SU(2)$. They are

$$A \equiv A_\mu dx^\mu, \Psi^\alpha \equiv \Psi_\mu^\alpha dx^\mu, \chi_\alpha^I, \eta_\alpha, \Phi_i, h^I, T \equiv T_\mu dx^\mu, H^I$$

The field s^α and δ^α transformations are obtained by imposing the equivariant commutation relations.

The invariant action cab be expressed as a $s\delta$ -exact term, and the invariance under the reduced symmetry with 6 parameters is big enough to determine it completely.

The auxiliary fields T and H show up naturally when solving the algebra and building the action.

$$\begin{aligned}
s_{(c)}^\alpha A &= \Psi^\alpha \\
s_{(c)}^\alpha \Psi_\beta &= \delta_\beta^\alpha T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i & s_{(c)}^\alpha h^I &= \chi^{\alpha I} \\
s_{(c)}^\alpha \Phi_i &= \frac{1}{2} \sigma_i{}^{\alpha\beta} \eta_\beta & s_{(c)}^\alpha \chi_\beta^I &= \delta_\beta^\alpha H^I + \sigma^i{}_\beta{}^\alpha [\Phi_i, h^I] \\
s_{(c)}^\alpha \eta_\beta &= -2\sigma^{ij}{}_\beta{}^\alpha [\Phi_i, \Phi_j] & s_{(c)}^\alpha H^I &= \frac{1}{2} [\eta^\alpha, h^I] + \sigma^{i\alpha\beta} [\Phi_i, \chi_\beta^I] \\
s_{(c)}^\alpha T &= \frac{1}{2} d_A \eta^\alpha + \sigma^{i\alpha\beta} [\Phi_i, \Psi_\beta]
\end{aligned}$$

$$\begin{aligned}
\delta_{(c)}^\alpha A &= g(\kappa) \eta^\alpha + g(J_I \kappa) \chi^{\alpha I} \\
\delta_{(c)}^\alpha \Psi_\beta &= \delta_\beta^\alpha (i_\kappa F - g(J_I \kappa) H^I) + \sigma^i{}_\beta{}^\alpha g(J_I \kappa) [\Phi_i, h^I] - 2\sigma^{ij}{}_\beta{}^\alpha g(\kappa) [\Phi_i, \Phi_j] \\
\delta_{(c)}^\alpha \Phi_i &= -\frac{1}{2} \sigma_i{}^{\alpha\beta} i_\kappa \Psi_\beta \\
\delta_{(c)}^\alpha \eta_\beta &= -\delta_\beta^\alpha i_\kappa T + \sigma^i{}_\beta{}^\alpha i_\kappa d_A \Phi_i \\
\delta_{(c)}^\alpha T &= \frac{1}{2} d_A i_\kappa \Psi^\alpha - g(J_I \kappa) ([\eta^\alpha, h^I] + \sigma^{i\alpha\beta} [\Phi_i, \chi_\beta^I]) + g(\kappa) \sigma^{i\alpha\beta} [\Phi_i, \eta_\beta] - \mathcal{L}_\kappa \Psi^\alpha \\
\delta_{(c)}^\alpha h^I &= -i_{J_I \kappa} \Psi^\alpha \\
\delta_{(c)}^\alpha \chi_\beta^I &= \delta_\beta^\alpha (i_\kappa d_A h^I + i_{J_I \kappa} T) + \sigma^i{}_\beta{}^\alpha i_{J_I \kappa} d_A \Phi_i \\
\delta_{(c)}^\alpha H^I &= \frac{1}{2} [i_\kappa \Psi^\alpha, h^I] + i_{J_I \kappa} d_A \eta^\alpha + \sigma^{i\alpha\beta} [\Phi_i, i_{J_I \kappa} \Psi_\beta] - i_\kappa d_A \chi^{\alpha I}
\end{aligned}$$

The action

The N=4 action is determined from $s, \bar{\sigma}, \delta$ invariance. Moreover, it can be written as

$$I_{N=4} = \int \frac{1}{|k|} s \delta \left[g(k) (AdA + \frac{2}{3} A^3) + g(J^I k)^* \epsilon_{IJK} h^J dh^K + \bar{s} \bar{\delta} \left(\frac{1}{2} h^I h_I - \frac{2}{3} \Phi_i \Phi^i \right) \right]$$

where $g(k) = g_{\mu\nu} k^\mu dx^\nu$ and $g(J^I k) = J^I_{\mu\nu} k^\mu dx^\nu$.

One has an even more symmetrical expression of the action

$$S_{N=4} = -\frac{1}{2} \int_M \text{Tr} F \wedge F + s_{(c)}^\alpha \delta_{(c)\alpha} \mathcal{G}$$

where the parameter of the δ symmetry is the constant κ and

$$\mathcal{G} = \int_M \text{Tr} \left(-\frac{1}{2} g(\kappa) \wedge (AF - \frac{1}{3} A^3) - \frac{1}{2} \star \epsilon_{IJK} h^I i_{J^J \kappa} d_A h^K + \star s_{(c)}^\alpha \delta_{(c)\alpha} \left(\frac{1}{2} h_I h^I - \frac{2}{3} \Phi^i \Phi_i \right) \right)$$

SUPERSYMMETRIC OBSERVABLES

There are supersymmetric loops, which are invariant under part of Poincare susy. This can be seen easily in the third twist. **The thirdly twisted representation can be obtained from the first one by the following mere field redefinitions**

$$V_\mu \equiv \kappa^\nu (h_{\mu\nu^-} + g_{\mu\nu} L)$$

$$\tilde{\Psi}_\mu \equiv \kappa^\nu (\bar{\chi}_{\mu\nu^-} + g_{\mu\nu} \bar{\eta})$$

$$\bar{\Psi}_\mu \equiv \kappa^\nu (\chi_{\mu\nu^+} + g_{\mu\nu} \tilde{\eta})$$

The parameters κ disappears, after doing all changes of variables

The set of fields in the third twist is thus

$$A_\mu, \Phi_i, h^I, \Psi_\mu^\alpha, \eta^\alpha, \chi^{I\alpha}, T_\mu, G^I \rightarrow A_\mu, V_\mu, \Phi, \bar{\Phi}, \Psi_\mu, \tilde{\Psi}_\mu, \chi_{\mu\nu^\pm}, \eta, \tilde{\eta}, H_{\mu\nu^\pm}, H$$

It has only an internal $U(1) \times SO(4)$ symmetry. One has a Q symmetry governed by two parameters u and v that satisfies the complex equation

$$(Q + d)(A + iV + c) + (A + iV + c) = F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)\Phi$$

By defining

$$Q\chi_{\pm} = H_{\pm} - [c, \chi_{\pm}]$$

one finds that the $N = 4$ action can be computed as a Q -exact term

$$I = \int d^4x \frac{1}{u^2 + v^2} Q \text{Re} [\chi_- + i\chi_+] (u + iv) (F_{A+iV} + H_- + iH_+) + \dots]$$

The action is independent on u and v . One can set $u = iv$. For this restriction of the parameters, the Q -transformation for A and V is such that

$$Q(A + iV) = -Dc$$

therefore the Wilson loop with argument $A + iV$ is Q -invariant

$$Q \exp \int dx^{\mu} (A_{\mu} + iV_{\mu}) = 0$$

and defines a (partially) supersymmetric observables.

These supersymmetric Wilson loops can be also expressed in the first twist formulation, since it is related to the third twist by a mere κ -dependent change of variables that leave invariant the action. One has thus

$$Q \exp \int_{\Gamma} dx^{\mu} (A_{\mu} + i\kappa^{\nu} (h_{\mu\nu} + g_{\mu\nu} L)) = 0$$

or

$$Q \exp \int_{\Gamma} dx^{\mu} (A_{\mu} + i\kappa^{\nu} (J_{\mu\nu}^J h^I + g_{\mu\nu} L)) = 0$$

Because the action is Q -exact, the mean value

$$\langle \exp \int_{\Gamma} dx^{\mu} (A_{\mu} + i\kappa^{\nu} (J_{\mu\nu}^J h^I + g_{\mu\nu} L)) \rangle$$

is independent on all possible local deformations, in particular those of the contour Γ . It is a sort of link number.

The results of stochastic quantization seem to say that, if the contour is 3d, one can use the 3d action

$$\int d^3x (AdA + 2/3A^3 + * \epsilon_{IJK} h^I D^J h^K + * LD_I h^I)$$

This 3-dimensional action can be interpreted as a complex Chern–Simons action.

Non renormalisation theorems

One may question finiteness, using Ward identities for the reduced symmetry that determines the action. Thanks to the closure and the possibility of getting rid of gauge transformation by using shadow fields, one proves finiteness for the beta function as well as for the $1/2$ BPS operators, using **standards arguments of locality and Ward identities**. In fact, the introduction of new fields called shadows, permit the control the possible breaking of supersymmetry that is induced by the gauge-fixing terms.

SHIFTING TO THE SPECIAL SUPERSYMMETRY OF THE CONFORMAL SECTOR

One has for the conformal susy 32 generators, with parameters ϵ and η ,

$$\begin{aligned}\delta A_\mu &= \lambda \Gamma_\mu (\epsilon + x^\mu \gamma_\mu \eta) \\ \delta \vec{\varphi} &= (\epsilon + x^\mu \gamma_\mu \eta) \vec{\tau} \lambda \\ \delta \lambda &= (\Gamma^{\mu\nu} F_{\mu\nu} + i D_\mu \Gamma^\mu \varphi - [\varphi, \varphi]) (\epsilon + x^\mu \gamma_\mu \eta) + 2i \varphi \eta\end{aligned}$$

The idea is of replacing $\kappa^\mu \rightarrow x^\mu$, and to see if we can distort the twisted N=4 algebra into another one that involves 4 graded scalar equivariant differential operators for ordinary and special susy $s_{(c)}$ and $\delta_{(c)}$, which are in involution as follows

$$\begin{aligned}\{s_{(c)}^\alpha, s_{(c)}^\beta\} &= 2\sigma^{i\alpha\beta} \delta_{\text{gauge}}(\Phi_i) \\ \{s_{(c)}^\alpha, \delta_{(c)}^\beta\} &= \varepsilon^{\alpha\beta} (\mathcal{L}_x + \delta_{\text{gauge}}(i_x A) + \mathbf{W}) + \Delta_{\mathbf{SL}(2, \mathbf{R})}^{\alpha\beta} \\ \{\delta_{(c)}^\alpha, \delta_{(c)}^\beta\} &= 2\sigma^{i\alpha\beta} \delta_{\text{gauge}}(|x|^2 \Phi_i)\end{aligned}$$

$\Delta_{SL(2, R)}$ is a $SL(2, R)$ transformation and W a $U(1)$ symmetry.

$$\begin{aligned}
s_{(c)}^\alpha A &= \Psi^\alpha \\
s_{(c)}^\alpha \Psi_\beta &= \delta_\beta^\alpha T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i & s_{(c)}^\alpha h^I &= \chi^{\alpha I} \\
s_{(c)}^\alpha \Phi_i &= \frac{1}{2} \sigma_i{}^{\alpha\beta} \eta_\beta & s_{(c)}^\alpha \chi_\beta^I &= \delta_\beta^\alpha H^I + \sigma^i{}_\beta{}^\alpha [\Phi_i, h^I] \\
s_{(c)}^\alpha \eta_\beta &= -2\sigma^{ij}{}_\beta{}^\alpha [\Phi_i, \Phi_j] & s_{(c)}^\alpha H^I &= \frac{1}{2} [\eta^\alpha, h^I] + \sigma^{i\alpha\beta} [\Phi_i, \chi_\beta^I] \\
s_{(c)}^\alpha T &= \frac{1}{2} d_A \eta^\alpha + \sigma^{i\alpha\beta} [\Phi_i, \Psi_\beta]
\end{aligned}$$

$$\begin{aligned}
\delta_{(c)}^\alpha A &= g(x) \eta^\alpha + g(J_I x) \chi^{\alpha I} \\
\delta_{(c)}^\alpha \Psi_\beta &= \delta_\beta^\alpha (i_x F - g(J_I x) H^I) + \sigma^i{}_\beta{}^\alpha g(J_I x) [\Phi_i, h^I] - 2\sigma^{ij}{}_\beta{}^\alpha g(x) [\Phi_i, \Phi_j] \\
\delta_{(c)}^\alpha \Phi_i &= -\frac{1}{2} \sigma_i{}^{\alpha\beta} i_x \Psi_\beta \\
\delta_{(c)}^\alpha \eta_\beta &= -\delta_\beta^\alpha i_x T + \sigma^i{}_\beta{}^\alpha i_x d_A \Phi_i + \mathbf{y} \sigma_i{}^{\alpha\beta} \Phi^i \\
\delta_{(c)}^\alpha T &= \frac{1}{2} d_A i_x \Psi^\alpha - g(J_I x) ([\eta^\alpha, h^I] + \sigma^{i\alpha\beta} [\Phi_i, \chi_\beta^I]) + g(x) \sigma^{i\alpha\beta} [\Phi_i, \eta_\beta] - \mathcal{L}_x \Psi^\alpha + \mathbf{z} \delta_\beta^\alpha \Psi^\beta \\
\delta_{(c)}^\alpha h^I &= -i_{J_I x} \Psi^\alpha \\
\delta_{(c)}^\alpha \chi_\beta^I &= \delta_\beta^\alpha (i_x d_A h^I + \mathbf{w} \mathbf{h}^I + i_{J_I x} T) + \sigma^i{}_\beta{}^\alpha i_{J_I x} d_A \Phi_i \\
\delta_{(c)}^\alpha H^I &= \frac{1}{2} [i_x \Psi^\alpha, h^I] + i_{J_I x} d_A \eta^\alpha + \sigma^{i\alpha\beta} [\Phi_i, i_{J_I x} \Psi_\beta] - i_x d_A \chi^{\alpha I} + \mathbf{w}' \chi^{\mathbf{I}\alpha}
\end{aligned}$$

The constants w, w', y, z are to be determined by imposing the anticommutation relations. One gets $z = 1, y = 2, w = 1, w' = 2$.

The change is by a non homogenous term in x for the δ^a transformations, and the special delta transformations can be written as

$$\delta = x^\mu (\delta_\mu + g_{\mu\nu} \frac{x^\nu}{x^2} C)$$

The δ_μ generator is identical to that of the vector supersymmetry of the superPoincaré algebra, and the operation C is the modification brought by the special supersymmetry, as it is implied by the graded commutation relations. The action of C is only non zero for $\eta^\alpha, \chi^{I\alpha}, T$ and H^I . So it seems that one can capture an interesting part of the maximal conformal supersymmetry with its 32 generators. However, I have not verified that the $N = 4$ action can be written as :

$$S_{N=4} = -\frac{1}{2} \int_M \text{Tr} F \wedge F + s_{(c)}^\alpha \delta_{(c)\alpha} \mathcal{G}$$

that is, it is invariant under scalar and special supersymmetry, as a $s\delta$ -exact term.

Special observables

Does it help to discuss special supersymmetric observables?

One can define the scalar differential, analogously as in Kapustin–Witten,

$$Q = u_\alpha s^\alpha + v_\alpha \delta^\alpha$$

where the $SL(2, R)$ -valued scalar parameters u_α, v_α are commuting and parametrize a supersymmetry with four parameters that is a twisted scalar sector of the conformal supersymmetry.

One has indeed that the following 1-form :

$$A + \frac{1}{x^2}(i_{J^J x} h^I + L) = dx^\mu \left(A_\mu + \frac{x^\nu}{x^2} (J_{\mu\nu}^J h^I + g_{\mu\nu} L) \right)$$

which transforms under $s + \bar{\delta}$ simply by a gauge transformation ($s + \bar{\delta}$ -invariance is just Q -invariance for the family of parameters defined by $u^\alpha = -v^\alpha$), and therefore the following special Wilson loop is $s + \bar{\delta}$ invariant :

$$(s + \bar{\delta}) \exp i \int_\Gamma \left(A + \frac{1}{x^2} (i_{J^J x} h^I + L) \right) = 0$$

This to be paralleled to the ordinary Wilson loop that is supersymmetric under a combination of the 2 scalar supersymmetry of the third twist formulation and can be expressed as follows :

$$\exp i \int_{\Gamma} (A + iV)$$

Do we have non renormalisation theorems for special observables ??

One may question the finiteness, independence with variations of contours, topological properties, etc.. of special observables that are supersymmetric, eg invariant under $s + \bar{\delta}$, and computed by mean of the $N = 4$ action that we have shown is s and $\bar{\delta}$ invariant and thus $s + \bar{\delta}$ -invariant. However, the use of Ward identities is complicated by the lack of translational invariance, due to the dependance on x of the transformations, so the topological properties are lost, and the observables probably depend on the detail of the contour.

References

- [1] L. Baulieu and G. Bossard, “New results on $N = 4$ superYang-Mills theory,” Phys. Lett. B **632**, 131 (2006) [arXiv:hep-th/0507003].
- [2] N. Berkovits, “A Ten-dimensional superYang-Mills action with off-shell supersymmetry,” Phys. Lett. B **318**, 104 (1993) [arXiv:hep-th/9308128]; L. Baulieu, N. Berkovits, G. Bossard and A. Martin, Ten-dimensional super-Yang–Mills with nine off-shell supersymmetries, Phys. Lett. B658 (2008) 249 [hep-th/0705.2002]; L. Baulieu, G. Bossard and A. Martin, “Twisted superspace,” Phys. Lett. B **658** (2008) 249, 0705.2002 [hep-th].