Physics 12c: Problem Set 3

Due: Thursday, April 25, 2019

1. Problem 8 in chapter 4 of Kittel and Kroemer
2. Problem 18 in chapter 4 of Kittel and Kroemer
3. **Relativistic ideal gas**

   Consider an ideal gas of N massless indistinguishable particles in a cubical box of side length L at temperature τ. A massless particle has energy $E = pc$, where $p = |\mathbf{p}|$ is the magnitude of the momentum and c is the speed of light. Assume that the number of particles is conserved (so this is different from a gas of photons).

   (a) Compute the single particle partition function $Z_1(\tau)$. Assume that the size of the box is large compared to the typical de Broglie wavelength of the particles.

   (b) Compute the N-particle partition function $Z_N(\tau)$. Assume that the gas is sufficiently dilute that no orbitals are multiply-occupied. Use Stirling's approximation if necessary to simplify your answer.

   (c) Compute the free energy and use it to obtain the entropy and pressure of the gas.

4. **Quantum and thermal noise in a harmonic oscillator**

   You may recall from Ph 12b that the position x and momentum p of a harmonic oscillator with circular frequency ω and mass m may be expressed as

   $$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad p = -\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger), \tag{1}$$

   where $[a, a^\dagger] = 1$. Thus $[x, p] = i\hbar$, and the Hamiltonian is

   $$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{2}$$

   It is convenient to describe the dynamics of the oscillator using the *Heisenberg picture* operator

   $$x(t) = e^{iHt/\hbar} x e^{-iHt/\hbar} \tag{3}$$

   $$= \sqrt{\frac{\hbar}{2m\omega}}\left(ae^{-i\omega t} + a^\dagger e^{i\omega t}\right). \tag{4}$$

   The expectation value of $x(t)x(0)$ is said to be the *time correlation function* for the oscillator's position; it quantifies how the position at time t is correlated with the position at time 0, and hence characterizes the fluctuations in position.

(a) Let $|n\rangle$ denote the n-th excited state of the oscillator, with energy $\hbar\omega(n+\frac{1}{2})$. Evaluate the expectation value

$$\langle n|x(t)x(0)|n\rangle. \tag{5}$$

(b) In the thermal ensemble at temperature τ, the probability that the oscillator's state is $|n\rangle$ is proportional to the Boltzmann factor $e^{-n\hbar\omega/\tau}$. The expectation value $\Delta_\tau(t)$ of $x(t)x(0)$ in this thermal ensemble has the form

$$\Delta_\tau(t) = \langle x(t)x(0)\rangle_\tau = P_\tau(\omega)e^{-i\omega t} + N_\tau(\omega)e^{i\omega t}, \tag{6}$$

where $P_\tau(\omega)$ and $N_\tau(\omega)$ are functions of ω. We say that $P_\tau(\omega)e^{-i\omega t}$ is the "positive frequency" part of $\Delta_\tau(t)$ and $N_\tau(\omega)e^{i\omega t}$ is the "negative frequency" part.

Find $P_\tau(\omega)$ and $N_\tau(\omega)$. Show that

$$\frac{N_\tau(\omega)}{P_\tau(\omega)} = e^{-\hbar\omega/\tau} \tag{7}$$

The *Kubo-Martin-Schwinger (KMS) condition* is a general property of fluctuations in thermal equilibrium at temperature τ, saying that fluctuations at negative frequency are suppressed relative to fluctuations at positive frequency by the Boltzmann factor $e^{-\hbar\omega/\tau}$. You have verified the KMS condition for the special case of the position of a harmonic oscillator.

(c) Evaluate $\Delta_\tau(t)$ in the limit $\tau \to 0$ and in the limit $\tau \to \infty$. Check that in the high-temperature limit, the correlation function does not depend on $\hbar$ — it describes "classical" thermal fluctuations of the oscillator.

(d) Prove the KMS condition more abstractly as follows. Recall that we can write

$$\Delta_\tau(t) = \langle x(t)x(0)\rangle_\tau = \mathrm{Tr}(e^{-\beta H}x(t)x(0)), \tag{8}$$

where $\beta = 1/\tau$ and $e^{-\beta H}$ is the density matrix in the canonical ensemble. By plugging in the definition (3) of $x(t)$ and using cyclicity of the trace, show that

$$\Delta_\tau(t - i\hbar\beta) = \Delta_\tau(-t). \tag{9}$$

Use this equation to rederive (7).

5. **Lifetime of a black hole**

A black hole with mass M (and hence energy Mc^2) has surface area $A = 4\pi R^2$, where $R = \frac{2GM}{c^2}$ is its "Schwarzschild radius." Its entropy is

$$\sigma = \frac{A}{4L_{\mathrm{Pl}}^2}, \tag{10}$$

where

$$L_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35}\text{m}, \tag{11}$$

is the "Planck length" (G is Newton's gravitational constant).

(a) Find the temperature τ of a block hole, expressed in terms of M, G, $\hbar$, and c.

(b) Black holes evaporate. Assuming the black hole radiates like a black body with temperature τ and surface area A, show that its mass $M(\tau)$ decreases as a function of time according to

$$M^2 \frac{dM}{dt} = -B, \tag{12}$$

where M is a constant. Express B in terms of $G, \hbar$, and c.

(c) By solving this differential equation, find the time t_M for a black hole to evaporate completely if its initial mass is M. Express t_M in terms of B and M.

(d) What is the lifetime of a solar mass black hole? Look up the mass of the supermassive black hole at the center of Messier 87, which was determined a week ago by the Event Horizon Telescope. What is the lifetime of that black hole (assuming that matter were to stop falling into it).