

## Physics 12c: Problem Set 4

Due: Thursday, May 2, 2019

1. Problem 12 in Chapter 5 of Kittel and Kroemer.
2. **Classical and quantum rigid rotor**

Consider a rod with moment of inertia  $I$ , rotating around a point. The configuration of the rod can be parameterized by the angles  $\theta, \phi$  in spherical coordinates. (We ignore rotations of the rod around its own axis.) The Hamiltonian is

$$H_{\text{rot}} = \frac{1}{2I} \left( p_{\theta}^2 + \frac{p_{\phi}^2}{\sin^2 \theta} \right), \quad (1)$$

where the momenta are  $p_{\theta} = I \frac{d\theta}{dt}$  and  $p_{\phi} = I \sin^2 \theta \frac{d\phi}{dt}$ .

- (a) Compute the classical partition function for the rod at temperature  $\tau$ .
- (b) Quantum mechanically, the states of the rod can be labeled by quantum numbers  $j, m$ , where  $m = -j, -j + 1, \dots, j - 1, j$ , and the energy is

$$E_{j,m} = \frac{\hbar^2}{2I} j(j+1). \quad (2)$$

Write down the quantum mechanical partition function. (Even though  $m$  doesn't affect the energy, don't forget to take it into account when counting states.) Take the high-temperature limit of the quantum mechanical partition function and show that you reproduce your answer from part (2a).

- (c) Plot the quantum and classical values of  $\langle E \rangle / \tau$  as a function of  $\tau$ . Don't worry about perfect accuracy — just make sure the shapes of the curves are clear.

### 3. Debye theory of capillary waves

Ripples of small amplitude and short wavelength on the surface of a liquid are called “capillary waves.” They are governed by surface tension  $\sigma$ , which has units of energy/area.

- (a) Given that the liquid has density  $\rho$  with units of mass/volume, use dimensional analysis to argue that capillary waves obey a dispersion relation

$$\omega^2 = Ck^3, \quad (3)$$

where  $\omega$  is the circular frequency,  $k$  is the wave number, and  $C$  is a constant. There is a single polarization for each wave vector.

- (b) Derive the analog of the Debye formula for the energy per unit surface area stored in the thermal surface vibrations in the limit of low temperature  $\tau$ . Express your answer in the form

$$U/A = \alpha C^\beta \hbar^\gamma \tau^\delta, \quad (4)$$

where  $U$  is energy,  $A$  is surface area, and  $\alpha, \beta, \gamma, \delta$  are numbers you are to find. As a check, verify the dimensional consistency of your answer. To evaluate  $\alpha$ , you might want to use the integral

$$I = \int_0^\infty \frac{x^{4/3} dx}{e^x - 1} = \Gamma\left(\frac{7}{3}\right) \zeta\left(\frac{7}{3}\right). \quad (5)$$

#### 4. Conservation laws and generalized potentials

Given any conserved quantity  $Q$ , we can consider a system  $\mathcal{S}$  and reservoir  $\mathcal{R}$  that are allowed to exchange  $Q$ . The usual argument for the Boltzmann distribution goes through and implies that the probability of a state  $s$  with  $Q = Q_s$  is proportional to  $P(s) \propto e^{-\alpha Q_s}$ , where  $\alpha = \frac{\partial \sigma_{\mathcal{R}}}{\partial Q}$ . We call  $\alpha$  a “generalized potential” for  $Q$ .

For multiple conserved quantities  $Q_i$ , the probability of a state  $s$  is proportional to  $P(s) \propto e^{-\sum_i \alpha_i Q_{i,s}}$ , where  $Q_{i,s}$  is the value of  $Q_i$  in the state  $s$  and  $\alpha_i$  are the generalized potentials for  $Q_i$ .

- (a) Consider a gas of classical particles moving in 1-dimension in a box that is sitting still at temperature  $\tau$ . Each particle has mass  $m$ . The probability distribution of velocities  $P(v)$  is controlled by conservation of energy — i.e. it is proportional to a Boltzmann factor  $e^{-E_s/\tau}$ . Compute  $P(v)$ .
- (b) Let us choose a new reference frame where the box is moving with velocity  $v_0$ . A particle that had velocity  $v$  in the old reference frame has velocity  $v' = v + v_0$  in the new reference frame. The probability distribution of velocities for particles in the box is still  $P(v) = P(v' - v_0)$ . Note that this is not a constant times  $e^{-E'_s/\tau}$ , where  $E'_s$  is the energy measured in the new frame. Explain why using an additional conserved quantity. What is the conserved quantity? What is the generalized potential for this quantity in the new reference frame?

#### 5. Isentropic model of the atmosphere

- (a) Consider an ideal gas of particles in the earth’s gravitational field, where each gas molecule has mass  $m$  and  $g$  is the acceleration due to gravity. The dependence of the pressure  $p(z)$  on the height  $z$  is determined by the condition for mechanical equilibrium: for the gas contained in a small region, the downward gravitational force is compensated by the difference between the pressure at the bottom of the region and the pressure at the top. Use this condition to express  $dp/dz$  in terms of  $m, g, p$ , and the temperature  $\tau$ .

- (b) Suppose the entropy per particle in the earth's atmosphere is independent of altitude, so that  $pn^{-\gamma}$  is constant, where  $n = N/V$  is the concentration, the number of gas molecules per unit volume. Use the ideal gas law  $\tau = p/n$  and the result of (5a) to show that  $d\tau/dz$  is a constant that can be expressed in terms of  $\gamma$ ,  $m$ , and  $g$ .
- (c) Assuming  $\gamma = 7/5$ , find  $dT/dz$  in  $^{\circ}\text{C}$  per km, where  $m$  is the mass of a nitrogen molecule. (Remember  $\tau = k_B T$ .)

## 6. Fluctuations in particle number

Using the Gibbs distribution, the expectation value for the particle number  $N$  of a system in thermal equilibrium and diffusive contact with a reservoir at temperature  $\tau$  and chemical potential  $\mu$  is

$$\langle N \rangle = \frac{\sum_s N_s e^{(N_s \mu - E_s)/\tau}}{\mathcal{Z}}$$

$$\mathcal{Z}(\tau, \mu) = \sum_s e^{(N_s \mu - E_s)/\tau}. \quad (6)$$

- (a) Show that

$$\langle (\Delta N)^2 \rangle = \langle (N - \langle N \rangle)^2 \rangle = \tau \left( \frac{\partial \langle N \rangle}{\partial \mu} \right)_{\tau}. \quad (7)$$

- (b) For a classical ideal gas, show that

$$\frac{\langle (\Delta N)^2 \rangle^{1/2}}{\langle N \rangle} = \frac{1}{\langle N \rangle^{1/2}}. \quad (8)$$

Hence, for  $N \gg 1$ , the fluctuations in particle number are small compared to the mean number of particles.