Physics 12c: Problem Set 5

Due: Thursday, May 16, 2019

1. **Higher order corrections to heat capacity**

   Consider a free gas of fermions with density of orbitals $\mathcal{D}(E) = \alpha E^{1/2}$ and Fermi energy E_F. The chemical potential and heat capacity have low-temperature expansions

$$\begin{aligned} \mu(\tau) &= E_F + \mu_2 \tau^2 + \mu_4 \tau^4 + \dots, \\ C_V(\tau) &= \gamma_1 \tau + \gamma_3 \tau^3 + \dots . \end{aligned} \tag{1}$$

   In class, we derived

$$\mu_2 = -\frac{\pi^2}{12 E_F}, \qquad \gamma_1 = \frac{\pi^2}{3} \alpha E_F^{1/2}. \tag{2}$$

   Find μ_4 and γ_3.

2. **(More) sharply-peaked functions and the path integral**

   The Hamiltonian for a particle moving in 1 dimension in a potential $V(x)$ is

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + V(\widehat{x}), \qquad \text{where} \qquad \widehat{p}|x\rangle = -i\hbar \frac{\partial}{\partial x}|x\rangle. \tag{3}$$

   We claim that for infinitesimal ϵ, we have

$$\langle x_1 | \left(1 - \frac{\epsilon \widehat{H}}{\hbar} + O(\epsilon^2) \right) | x_0 \rangle = A_\epsilon \exp\left(-\frac{1}{\hbar} \int_0^\epsilon dt\, L(x(t), \dot{x}(t)) \right), \tag{4}$$

   where $A_\epsilon = \sqrt{\frac{m}{2\pi\hbar\epsilon}}$ is a constant,

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 + V(x) \tag{5}$$

   is the (imaginary time) Lagrangian, and

$$x(t) = x_0 + (x_1 - x_0)\frac{t}{\epsilon} \tag{6}$$

   is a straight-line path from x_0 to x_1 taking time ϵ. Above, $O(\epsilon^2)$ means quadratic order and higher in ϵ.

   (a) Prove (4) as follows. Compute the left-hand side in terms of $\delta(x_1 - x_0)$ and $\delta''(x_1 - x_0)$. Then plug (5) and (6) into the right-hand side. Expand the answer to linear order in ϵ and check that it agrees with the left-hand side. *Hint:* use the same techniques as in problem (1).

(b) Show that

$$\langle x_N| \left(1 - \frac{T\widehat{H}}{\hbar N} + O\left(\frac{1}{N^2}\right)\right)^N |x_0\rangle$$
$$= A_{\frac{T}{N}} \int_{-\infty}^{\infty} A_{\frac{T}{N}} dx_1 \ldots \int_{-\infty}^{\infty} A_{\frac{T}{N}} dx_{N-1} \exp\left(-\frac{1}{\hbar}\int_0^T dt L(x(t), \dot{x}(t))\right), \tag{7}$$

where $x(t)$ is a piecewise-linear path between the values

$$x(0) = x_0, \quad x(\tfrac{T}{N}) = x_1, \quad x(\tfrac{2T}{N}) = x_2, \quad \ldots \quad x(T) = x_N. \tag{8}$$

(c) Show that

$$\mathrm{Tr}\left[\left(1 - \frac{T\widehat{H}}{\hbar N} + O\left(\frac{1}{N^2}\right)\right)^N\right]$$
$$= \int_{-\infty}^{\infty} A_{\frac{T}{N}} dx_0 \ldots \int_{-\infty}^{\infty} A_{\frac{T}{N}} dx_{N-1} \exp\left(-\frac{1}{\hbar}\int_0^T dt L(x(t), \dot{x}(t))\right), \tag{9}$$

where $x(t)$ is a piecewise-linear periodic path between the values

$$x(0) = x_0, \quad x(\tfrac{T}{N}) = x_1, \quad x(\tfrac{2T}{N}) = x_2, \quad \ldots \quad x(T) = x(0). \tag{10}$$

That's the end of this problem; now here's some fun information. Taking the limit $N \to \infty$ of (7), we obtain the path integral

$$\langle x_f|e^{-\frac{T\widehat{H}}{\hbar}}|x_i\rangle = \int_{\substack{x(0)=x_i \\ x(T)=x_f}} \mathcal{D}x(t) \exp\left(-\frac{1}{\hbar}\int_0^T dt L(x(t), \dot{x}(t))\right). \tag{11}$$

Here, the integral is over paths from $x(0) = x_i$ to $x(T) = x_f$. One way of defining the measure $\int \mathcal{D}x(t)$ on the space of paths is by approximating the path as piecewise linear and integrating over the intermediate positions. In other words, (11) is just a fancy way of writing (7). Note that $e^{-i\Delta t\widehat{H}/\hbar}$ is the operator that evolves a state forward by time Δt, so $e^{-T\widehat{H}/\hbar}$ is the operator that evolves by imaginary time $\Delta t = -iT$. Thus, the path $x(t)$ should be interpreted as being a path in imaginary time.

Taking the limit $N \to \infty$ of (9), we get

$$\mathrm{Tr}(e^{-\frac{T\widehat{H}}{\hbar}}) = \int_{x(0)=x(T)} \mathcal{D}x(t) \exp\left(-\frac{1}{\hbar}\int_0^T dt L(x(t), \dot{x}(t))\right), \tag{12}$$

where here the integral is over periodic paths. Note that the left-hand side is the partition function at inverse temperature $\beta = 1/\tau = T/\hbar$. Thus, the partition function is a path integral over paths that are periodic in imaginary time with periodicity $T = \hbar\beta$, or $\Delta t = -i\hbar\beta$.

3. **Heat capacity of graphene**

Geim and Novoselov received the 2010 Nobel Prize in Physics for their studies of graphene, a single layer of carbon atoms bonded into a two-dimensional hexagonal lattice. Remarkably, electrons in graphene behave like relativistic massless fermions; for each value of the wavenumber $\mathbf{k} = (k_x, k_y)$, there are two single-particle orbitals, with energies

$$E_\pm(\mathbf{k}) = \pm\hbar v|\mathbf{k}|. \tag{13}$$

The Fermi energy is $E_F = 0$; hence at zero temperature the orbitals with negative energy are occupied, and the orbitals with positive energy are empty.

Assuming the electrons can be treated as an ideal gas, and that there are two spin states for each orbital, the internal energy of the electrons has the form

$$U(\tau) - U(0) = \frac{1}{3}\gamma A\tau^3, \tag{14}$$

where A denotes the area and hence the electron heat capacity is $C = \gamma A\tau^2$. Find γ. *Hint:* Do not use approximations of $f(E, \tau, \mu)$ in terms of δ-functions and their derivatives because the density of states is not differentiable at $E = E_F$. Instead, use the integral $\int_0^\infty dx\, x^2/(e^x + 1) = 3\zeta(3)/2$.

4. **Bose condensation in two dimensions**

Consider an ideal gas of non-relativistic spin-0 bosons, at temperature τ, in a *two-dimensional* box of side length L.

(a) Find the *two-dimensional* density of orbitals $\mathcal{D}(E)$.

(b) Express the chemical potential μ in terms of $N_0(\tau)$, the number of particles in the ground orbital. Use the conventions that the energy of the ground orbital is $E_0 = 0$.

(c) Find $N_e(\tau)$, the number of particles in excited orbitals. You may assume that the box is big enough so that the sum over modes can be replaced by an integral. Be sure to use the formula found in (4b) for μ, not the $N_0 \to \infty$ limit of that formula. Your answer for N_e will therefore be expressed in terms of N_0. *Hint:* $\int dx(ae^x - 1)^{-1} = \log(a - e^{-x})$.

(d) Find the *two-dimensional* Einstein condensation temperature τ_E. This is the smallest temperature such that for $\tau > \tau_E$, the fraction $N_0/(N_0+N_e)$ of particles in the ground orbital vanishes in the limit $L \to \infty$. (The limit is to be taken with the density $(N_0 + N_e)/L^2$ held fixed.)