Physics 12c: Problem Set 6

Due: Thursday, May 23, 2019

1. **Light bulb in a refrigerator**

   A refrigerator that draws 50 W of power is contained in a room at temperature 300°K. A 100 W lightbulb is left burning inside the refrigerator. Find the steady-state temperature inside the refrigerator assuming it operates reversibly and is perfectly insulated.

2. **Photonic heat engine**

   Consider a heat engine undergoing a Carnot cycle, where the working fluid is a photon gas rather than a classical ideal gas. In the first stage, the gas expands isothermally at temperature τ_h from the initial volume V_1 to the final volume V_2. In the second stage, it expands isentropically to volume V_3, cooling to temperature τ_l. In the third stage it is compressed isothermally at temperature τ_l to volume V_4, and in the fourth stage it is compressed isentropically back to volume V_1, heating to temperature τ_h.

   (a) The energy per unit volume of a photon gas is $U/V = A\tau^4$, where $A = \pi^2/15\hbar^3c^3$. Use the thermodynamic identity

   $$dU = \tau d\sigma - pdV \tag{1}$$

   to find the entropy of the gas, expressed in terms of A, τ, and V. Assume that the entropy is zero at $\tau = 0$.

   (b) Use the thermodynamic identity again to express the pressure p in terms of A, τ, and V.

   (c) Calculate the work done W_{12} and the heat added Q_{12} during the first stage of the cycle, expressed in terms of A, τ_h, V_1, and V_2. Verify that $Q_{12} - W_{12}$ is the change in the internal energy of the gas.

   (d) Express the work W_{34} done by the gas in the third stage in terms of A, τ_l, V_3, and V_4.

   (e) Use the condition σ =constant during the isentropic stages to express V_3 and V_4 in terms of τ_h, τ_l, V_1, and V_2.

   (f) Find the work W_{23} done during the second stage and the work W_{41} done during the fourth stage.

   (g) Express the net work W done during the complete cycle in terms of A, τ_h, τ_l, V_1, and V_2. Comparing to Q_{12}, check that the engine achieves the ideal Carnot efficiency.

3. **Speed vs. reversibility**

Consider two systems $\mathcal{S}_1, \mathcal{S}_2$ with temperatures τ_1, τ_2. Between the systems, we place a barrier with a finite thermal conductivity $\alpha > 0$, so that the rate of energy flow is

$$\frac{dU_1}{dt} = \alpha(\tau_2 - \tau_1). \tag{2}$$

Let $\mathcal{S}_1$ have constant heat capacity $\frac{dU_1}{d\tau_1} = C$.

Suppose that we can set τ_2 to any function of time $\tau_2(t)$. Furthermore, we can do so without producing entropy anywhere other than the interface between $\mathcal{S}_1$ and $\mathcal{S}_2$. For example, we can imagine that $\mathcal{S}_2$ is a very large insulated container of gas that we can expand or compress at will with a piston.

The goal of this problem is to show that entropy production at the interface between $\mathcal{S}_1$ and $\mathcal{S}_2$ can be suppressed by changing τ_2 very slowly.

(a) Show that the rate of change of the total entropy $\sigma_{\text{tot}} = \sigma_1 + \sigma_2$ due to heat exchange through the barrier is

$$\frac{d\sigma_{\text{tot}}}{dt} = \alpha\frac{(\tau_2 - \tau_1)^2}{\tau_1\tau_2}. \tag{3}$$

(b) Define the temperature difference $\tau_1(t) - \tau_2(t) = \delta(t)$. Show that it satisfies the differential equation

$$\frac{d\delta}{dt} = -\gamma\delta - \dot{\tau}_2, \tag{4}$$

where $\dot{\tau}_2 = \frac{\partial \tau_2}{\partial t}$ and $\gamma = \frac{\alpha}{C}$. Assuming that $\dot{\tau}_2$ is zero in the far past and far future, show that the solution is

$$\delta(t) = -\int_{-\infty}^{t} e^{\gamma(t'-t)}\dot{\tau}_2(t')dt'. \tag{5}$$

(c) Let us compute $\delta(t)$ for a slow process. Note that $1/\gamma$ is a time scale associated with heat transfer. For a slow process, this time scale is much shorter than the time scale of changes in τ_2. We can equivalently think of this as the limit $\gamma \to \infty$.

To simplify (5) in this limit, first write it as

$$\delta(t) = -\int_{-\infty}^{\infty} K(t-t')\dot{\tau}_2(t')dt', \tag{6}$$

where

$$K(t) \equiv e^{-\gamma t}\theta(t). \tag{7}$$

and $\theta(x)$ is defined by

$$\theta(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0. \end{cases} \tag{8}$$

When $\gamma \to \infty$, the kernel $K(t)$ becomes very sharply-peaked. Approximate it as a delta function using the methods from the previous problem set. Show that

$$\delta(t) \approx -\dot{\tau}_2(t)/\gamma \qquad (\gamma \text{ large}). \tag{9}$$

Discussion: The result $\delta = -\dot{\tau}/\gamma$ is called "adiabatic elimination". The idea is that when γ is large, the term $-\gamma\delta$ causes the rate of change $\frac{\partial \delta}{\partial t}$ to be enormous unless the two terms on the right-hand side of (4) very nearly cancel. If they initially don't cancel, then δ will rapidly move to make them cancel. Thus, in the limit $\gamma \to \infty$, we can set the sum of the terms to zero $-\gamma\delta - \dot{\tau} = 0$. In the language of electronics, the term $-\gamma\delta$ creates strong "negative feedback." Adiabatic elimination is useful for understanding the behavior of op-amps in circuits.

(d) Consider changing τ_2 from an initial value τ_0 to a final value $\tau_0(1+x)$ such that $\dot{\tau}_2$ is a gaussian with width T:

$$\dot{\tau}_2 = \frac{\tau_0 x}{\sqrt{2\pi T^2}} \exp\left(-\frac{t^2}{2T^2}\right). \tag{10}$$

Using the value of $\delta(t)$ from problem (3c), and assuming $x \ll 1$, compute the total change in entropy $\sigma_{\text{tot}}(t = \infty) - \sigma_{\text{tot}}(t = -\infty)$. Show that the change in entropy can be made arbitrarily small by making the change in τ_2 happen arbitrarily slowly, $T \to \infty$.