Phys 229a, CFT: Problem Set 1

Due: February 1, 2018
Please write up your solutions in LaTeX, and submit via email (dsd@caltech.edu). Feel free to use a computer algebra program (e.g. *Mathematica*).

1. Consider the quantum harmonic oscillator on the Euclidean time interval $[\tau_a, \tau_b]$. This theory has Euclidean action

$$S[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 \right). \tag{1}$$

Inside a functional integral, we can integrate by parts to obtain

$$0 = \int Dx \frac{\delta}{\delta x(\tau)} (\cdots), \tag{2}$$

where $(\cdots)$ is any insertion.

(a) Consider a one-point function with fixed boundary conditions for the path integral,

$$G(\tau) = \frac{\int_{\substack{x(\tau_a)=x_a \\ x(\tau_b)=x_b}} Dx\, x(\tau) e^{-S[x]}}{\int_{\substack{x(\tau_a)=x_a \\ x(\tau_b)=x_b}} Dx\, e^{-S[x]}} = \frac{\langle x_b, \tau_b | \widehat{x}(\tau) | x_a, \tau_a \rangle}{\langle x_b, \tau_b | x_a, \tau_a \rangle}, \tag{3}$$

where $\tau \in [\tau_a, \tau_b]$. Here, $|x, \tau\rangle = e^{-\tau H}|x\rangle$. Use the integration by parts identity (2) to derive the equation of motion

$$(\partial_\tau^2 - \omega^2) G(\tau) = 0. \tag{4}$$

What are the boundary conditions for $G(\tau)$? Use these together with (4) to determine $G(\tau)$.

(b) Now consider a time-ordered two-point function

$$G(\tau_1, \tau_2) = \frac{\int_{\substack{x(\tau_a)=x_a \\ x(\tau_b)=x_b}} Dx\, x(\tau_1) x(\tau_2) e^{-S[x]}}{\int_{\substack{x(\tau_a)=x_a \\ x(\tau_b)=x_b}} Dx\, e^{-S[x]}} = \frac{\langle x_b, \tau_b | T\{\widehat{x}(\tau_1)\widehat{x}(\tau_2)\} | x_a, \tau_a \rangle}{\langle x_b, \tau_b | x_a, \tau_a \rangle}. \tag{5}$$

Derive the equation of motion

$$(\partial_{\tau_1}^2 - \omega^2) G(\tau_1, \tau_2) + \delta(\tau_1 - \tau_2) = 0. \tag{6}$$

What are the boundary conditions? Solve the equation of motion to derive $G(\tau_1, \tau_2)$.

(c) Take a limit $\tau_a \to -\infty, \tau_b \to +\infty$ to compute $\langle 0|T\{\widehat{x}(\tau_1)\widehat{x}(\tau_2)\}|0\rangle$ where $|0\rangle$ is the ground state. Compute the same time-ordered product using ladder operators, and check that they agree.

(d) Recall that we can obtain any Lorentzian correlator from a time-ordered Euclidean correlator by analytic continuation. If we have a Euclidean operator $\widehat{O}_E(\tau) = e^{\tau H}\widehat{O}_E(0)e^{-\tau H}$, then we are interested in correlators of Lorentzian operators $\widehat{O}_L(t) = e^{itH}\widehat{O}_E(0)e^{-itH} = O_E(\tau = it)$. Given a time-ordered Euclidean correlator ("Schwinger function") in some states

$$G(\tau_1, \cdots, \tau_n) = \langle \Psi'|T\{\widehat{O}_E(\tau_1)\cdots\widehat{O}_E(\tau_n)\}|\Psi\rangle, \tag{7}$$

the Lorentzian correlators ("Wightman functions") are given by

$$\langle \Psi'|\widehat{O}_L(t_1)\cdots\widehat{O}_L(t_n)|\Psi\rangle = \lim_{\epsilon_i \to 0} G(\epsilon_1 + it_1, \cdots, \epsilon_n + it_n), \tag{8}$$

where $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$. (This choice of the ϵ's ensures that the Euclidean operators are in the correct order to begin with. Then we only need to continue in the imaginary parts of their time coordinates.)

Given this prescription, compute the commutator

$$\frac{\langle x_b, \tau_b|[\widehat{x}_L(t_1), \widehat{x}_L(t_2)]|x_a, \tau_a\rangle}{\langle x_b, \tau_b|x_a, \tau_a\rangle}. \tag{9}$$

Show that if $\widehat{p}(t) = \frac{\partial}{\partial t}\widehat{x}_L(t)$, then

$$[\widehat{x}_L(t), \widehat{p}(t)] = i, \tag{10}$$

which is the usual canonical commutation relation.

(e) What would be different in the above calculation if we replaced the harmonic potential with a general potential $\frac{1}{2}\omega^2 x^2 \to V(x)$? (Don't do the whole calculation, just describe what would be different.) Argue that we'd get the same equal time commutation relation in the more general case.

(f) Read Appendix A of Polchinski, Volume 1 to understand how to compute the transition amplitudes $\langle x_b, \tau_b|x_a, \tau_a\rangle$.

2. A Euclidean two-point function of a scalar operator in a CFT has the form

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{1}{(x_1 - x_2)^{2\Delta}}, \tag{11}$$

where $x_1, x_2 \in \mathbb{R}^d$ and Δ is a real number. Let us write $x = (\tau, \mathbf{x})$ and consider the continuation to Lorentzian signature $\tau \to it$. We denote $x_L = (t, \mathbf{x}) \in \mathbb{R}^{1,d-1}$ (with signature $- + \cdots +$) and $\phi_L(x_L) = \phi(i\tau, \mathbf{x})$. What are

(a) $\langle 0|\phi_L(x_{1L})\phi_L(x_{2L})|0\rangle$?

(b) $\langle 0|T\{\phi_L(x_{1L})\phi_L(x_{2L})\}|0\rangle$? (here T denotes Lorentzian time-ordering)

(c) $\langle 0|[\phi_L(x_{1L}),\phi_L(x_{2L})]|0\rangle$?

Hint: there are three distinct configurations of two points in Lorentzian signature, and the answer may be different for each. Optional: Can you write compact expressions using an $i\epsilon$ prescription?

3. Consider the 1d Ising model with a next-to-nearest neighbor interaction

$$S[s] = -K\sum_i s_i s_{i+1} - K'\sum_i s_i s_{i+2} \tag{12}$$

Quantize the theory by finding a Hilbert space $\mathcal{H}$ and transfer matrix $\widehat{T}:\mathcal{H}\to\mathcal{H}$ so that the partition function on an appropriate lattice is $\mathrm{Tr}(\widehat{T}^N)$. What would you do in the case of a (next-to)^n-nearest neighbor interaction?