# Phys 229a, CFT: Problem Set 2

Due: February 20, 2018
Please write up your solutions in LaTeX, and submit via email (dsd@caltech.edu). Feel free to use a computer algebra program (e.g. *Mathematica*).

1. (From Bootstrap School 2017) Consider an XY or Heisenberg magnet whose magnetic ions are arranged in a cubic lattice. In this case, the interactions with the lattice break the $O(N)$ rotational group acting on the spin vectors $\vec{\phi} = (\phi_1, \cdots, \phi_N)$. Thus, additional terms appear in the Hamiltonian that are not $O(N)$-invariant. A typical example is given by the Euclidean action

$$S = \int d^d x \left( \sum_{i=1}^{N} \left( \frac{1}{2}(\partial\phi_i)^2 + t_0 \phi_i^2 \right) + u_0 \left( \sum_{n=1}^{N} \phi_i^2 \right)^2 + v_0 \sum_{i=1}^{N} \phi_i^4 \right), \tag{1}$$

where $t_0, u_0$, and $v_0$ are dimensionful coupling constants related to dimensionless couplings by $t = t_0 a^2$, $u = u_0 a^{4-d}$, and $v = v_0 a^{4-d}$. Here, $a$ is the UV cutoff. Let us assume $u + v > 0$ in order to ensure that the action is bounded from below.

We are interested in studying this model in $d = 3$ when $N = 2$ (XY model) or $N = 3$ (Heisenberg model). This is of course very hard, so let us study this model in the $4 - \epsilon$ expansion (for any $N$).

   (a) Which term in the Euclidean action breaks the $O(N)$ symmetry? What is the global symmetry group when $N = 2, 3$?

   (b) In a series expansion at small $u$ and $v$, the beta functions for the three coupling constants can be written as[1]

$$\begin{aligned}
-\beta_t &= a\frac{dt}{da} = c_1 t - 8(N+2)ut - 24vt + \ldots, \\
-\beta_u &= a\frac{du}{da} = c_2 u - 8(N+8)u^2 - 48uv + \ldots, \\
-\beta_v &= a\frac{dv}{da} = c_3 v - 96uv - 72v^2 + \ldots .
\end{aligned} \tag{2}$$

   What are the values of the numerical constants $c_1, c_2$, and $c_3$?

   (c) Find all the RG fixed points when $\epsilon \ll 1$. At each fixed point, calculate the scaling dimensions of the three operators that multiply $t, u$, and $v$, respectively in the action. Note that there might be mixing between these operators.

   (d) Based on the values of the scaling dimensions you determined, which fixed point is the most stable one? Is there a critical value of $N = N_c$ where the stability of the fixed points changes?

[1] Here, we have rescaled the couplings by some factors of $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ to make the $\beta$-functions simpler.

(e) Sketch an RG flow diagram.

2. Consider a QFT coupled to a background metric $g$. For concreteness, suppose correlators are given by the path integral

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g = \int D\phi\, \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\, e^{-S[g,\phi]}. \tag{3}$$

A stress tensor insertion is the response to a small metric perturbation,

$$\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g. \tag{4}$$

Derive the Ward identity

$$\partial_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = -\sum_i \delta(x-x_i)\partial_i^\nu \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \tag{5}$$

by demanding that $S[g,\phi]$ be diffeomorphism invariant near flat space. (A correlator $\langle\cdots\rangle$ without the $g$-subscript means a correlator in flat space $g_{\mu\nu}=\delta_{\mu\nu}$.) Assume that the operators $\mathcal{O}_i$ have been defined so that they transform as scalars under diffeomorphisms.[2] How should we modify (5) when the $\mathcal{O}_i$ have spin?

3. Consider the free scalar on a manifold with metric $g$, with a nontrivial curvature-dependent mass,

$$S_{\text{free}}[g,\phi] = \int d^dx\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\xi R\phi^2\right). \tag{6}$$

(a) Using the definition $T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}$, show that the stress tensor, evaluated in flat space, is given by

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}(\partial\phi)^2 - \xi(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^2. \tag{7}$$

(b) For what value of $\xi$ is $T^{\mu\nu}$ traceless in flat space? (You will have to use the equation of motion for $\phi$.)

(c) Consider a Weyl transformation where we rescale the metric and the field $\phi$ by a position-dependent factor

$$\begin{aligned} g_{\mu\nu}(x) &\to e^{2\omega(x)}g_{\mu\nu}(x) \\ \phi(x) &\to e^{-\Delta\omega(x)}\phi(x). \end{aligned} \tag{8}$$

[2] This is why this version of the Ward identity does not have the same $(\partial\epsilon)$ terms as in the notes. For example, we define the operator $(\partial\phi)^2$ as $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ in this formalism, and the variation of $g^{\mu\nu}$ cancels the term proportional to $\partial\epsilon$ in the notes.

Let $\xi$ be the value that makes the stress tensor traceless in flat space, computed in (3b). Show that for this value of $\xi$, the curved-space action $S_{\text{free}}[g, \phi]$ is Weyl-invariant, provided $\Delta$ is chosen appropriately. (Feel free to use formulae from your favorite GR/differential geometry book.)