01/08/2013

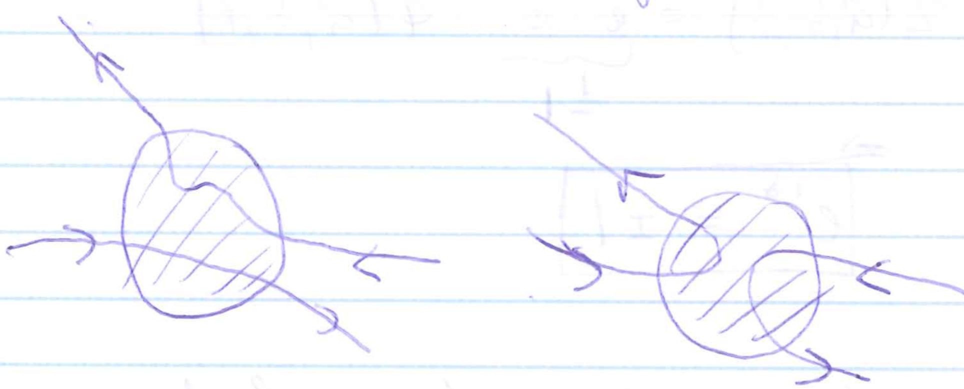
# Recitation 1 - covers ch. 10 of Phillips QM

## • identical particles and exchange statistics in QM

→ in classical physics : can keep track of individual particles

→ in QM : identical particles indistinguishable (identical)

in QM, we can not distinguish the 2 situations



## 2 particle system

particles $p$ & $q$

system described by 2-particle wavefunction

$\Psi(\vec{r}_p, \vec{r}_q, t)$

probability to find particle $p$ in a volume element $d^3a$ @ $\vec{r}_p = \vec{a}$
and " $q$ " " " " $d^3b$ @ $\vec{r}_q = \vec{b}$

$|\Psi(\vec{a}, \vec{b}, t)|^2 \, d^3a \, d^3b$

identical particles

$$|\Psi(\vec{a},\vec{b},t)|^2 = |\Psi(\vec{b},\vec{a},t)|^2$$

$$\Rightarrow \Psi(\vec{a},\vec{b},t) = e^{i\delta}\,\Psi(\vec{b},\vec{a},t)$$

exchange $\vec{a},\vec{b}$

$$\Psi(\vec{b},\vec{a},t) = e^{i\delta}\,\Psi(\vec{a},\vec{b},t)$$

$$\Rightarrow \Psi(\vec{a},\vec{b},t) = \underbrace{e^{i\delta}e^{i\delta}}_{\pm 1}\,\Psi(\vec{a},\vec{b},t)$$

$$\Rightarrow \boxed{e^{i\delta} = \pm 1}$$

$\Rightarrow$ symmetric or antisymmetric wavefunctions

$$\Psi(\vec{a},\vec{b},t) = \pm\,\Psi(\vec{b},\vec{a},t)$$

$+ \rightarrow$ boson

$- \rightarrow$ fermion

- permutation operator $\hat{P}$ $\quad \hat{P}|ab\rangle = \eta_p |ba\rangle$

$$\hat{P}^2|ab\rangle = \eta_p^2|ab\rangle \underset{\uparrow}{=} |ab\rangle$$

doing permutation twice gives the same state

$$\Rightarrow \eta_p^2 = 1$$

$$\Rightarrow \eta_p = \pm 1$$

# Exchange forces

1 particle in state $\psi_a(\cancel{x_1})$
1 particle in state $\psi_b(\cancel{x_2})$

→ 2 particle wave function: ~~$\psi(x_1 x_2)$~~

① distinguishable bosons
particle 1 in state $\psi_a$
particle 2 in state $\psi_b$

$$\rightarrow \psi(x_1, x_2) = \psi_a(x_1)\,\psi_b(x_2)$$

② identical bosons:

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\,\psi_b(x_2) + \psi_a(x_2)\,\psi_b(x_1)\right)$$

③ identical fermions:

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\,\psi_b(x_2) - \psi_a(x_2)\,\psi_b(x_1)\right)$$

calculate the square of the separation distance between 2 particles

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

case 1 for distinguishable particles

$$\langle x_1^2 \rangle = \int \psi_a^*(x_1)\, \psi_b^*(x_2)\, x_1^2\, \psi_a(x_1)\, \psi_b(x_2)\, d^3x_1\, d^3x_2$$

$$= \int x_1^2\, |\psi_a(x_1)|^2\, d^3x_1 \underbrace{\int |\psi_b(x_2)|^2\, dx_2^3}_{=1}$$

$$= \langle x^2 \rangle_a$$ (the expectation value of $x^2$ in the one particle state a)

$$\langle x_2 \rangle^2 = \dots = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \dots \quad \langle x \rangle_a \langle x \rangle_b$$

$$\Rightarrow \langle (x_1 - x_2)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b$$

$d$ ↑ distinguishable

case 2 indistinguishable particles:

$$\langle x_1^2 \rangle = \frac{1}{2} \int \left[ \left( \psi_a^*(x_1)\, \psi_b^*(x_2) \pm \psi_a^*(x_2)\, \psi_b^*(x_1) \right) x_1^2 \left( \psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1) \right) \right] d^3x_1\, d^3x_2$$

$$= \frac{1}{2} \Big[ \int x_1^2\, |\psi_a(x_1)|^2\, d^3x_1 \int d^3x_2\, |\psi_b(x_2)|^2 + \int x_1^2\, |\psi_b(x_1)|^2\, dx_1^2 \int |\psi_a(x_2)|^2$$

$$\pm \int x_1^2\, \psi_a^*(x_1)\, \psi_b(x_1)\, dx_1 \underbrace{\int dx_2^3\, \psi_b^*(x_2)\, \psi_a(x_2)}_{=0} \quad = \frac{1}{2}\left( \langle x^2 \rangle_a + \langle x^2 \rangle_b \right)$$

$\pm\; a \leftrightarrow b$

same for $\langle x_2^2 \rangle = \frac{1}{2} \left( \langle x^2 \rangle_a + \langle x^2 \rangle_b \right)$

but $\langle x_1 x_2 \rangle = \frac{1}{2} \Big[ \int x_1 |\psi_a(x_1)|^2 d^3x_1 \int x_2 |\psi_b(x_2)|^2 d^3x_2$

$+ \int x_1 |\psi_b(x_1)|^2 d^3x_1 \int x_2 |\psi_a(x_2)|^2 d^3x_2$

$\pm \int x_1 \psi_a^*(x_1) \psi_b(x_1) dx_1 \int x_2 \psi_b^*(x_2) \psi_a(x_2) d$

$\pm \quad a \leftrightarrow b$

$= \frac{1}{2} \Big( \langle x \rangle_a \langle x \rangle_b + \langle x \rangle_b \langle x \rangle_a \pm \langle x \rangle_{ab} \langle x \rangle_{ba} \pm \langle x \rangle_{ba} \langle x \rangle_{ab}$

$= \langle x \rangle_a \langle x \rangle_b \pm |\langle x \rangle_{ab}|^2$

$\langle x \rangle_{ab} = \int d^3x \, x \, \psi_a^*(x) \psi_b(x)$

$\Rightarrow \langle (x_1 - x_2)^2 \rangle_\pm = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b \mp 2 |\langle x \rangle_{ab}$

$= \langle (x_1 - x_2)^2 \rangle_d \underbrace{\mp 2 |\langle x \rangle_{ab}|^2}$

different in comparison to distinguishable particles interpretation:

- identical bosons somewhat closer together
  " fermions " farther apart

- $\langle x \rangle_{ab}$ vanishes unless wavefunctions overlap.

  $\Rightarrow$ if $\psi_a$ represents an $e^-$ in an atom in Chicago
  and $\psi_b$ " an $e^-$ in " " in Pasadena,
  then the overlap $\langle x \rangle_{ab}$ is zero
  → not going to make a difference, whether one symmetrizes the wavefunction or not.

- interesting case: when there is finite overlap between wavefunctions

  system behaves as if there were an attractive (repulsive) force for identical bosons (fermions)

  → strictly QM phenomenon

e.g. degeneration pressure in white dwarfs and neutron stars.

Problem 5.7 in Griffiths

3-particle state

$$\left.\begin{array}{l}\psi_a(x)\\ \psi_b(x)\\ \psi_c(x)\end{array}\right\}\text{ orthonormal states}$$

construct the 3-particle states for

(a) distinguishable particles:

$$\psi(x_1, x_2, x_3) = \psi_a(x_1)\,\psi_b(x_2)\,\psi_c(x_3)$$

(b) indistinguishable bosons:

$$\begin{aligned}\psi(x_1, x_2, x_3) = N\big[&\psi_a(x_1)\,\psi_b(x_2)\,\psi_c(x_3) + \psi_a(x_2)\,\psi_b(x_1)\,\psi_c\\ &+ \psi_a(x_3)\,\psi_b(x_2)\,\psi_c(x_1) + \psi_a(x_1)\,\psi_b(x_3)\,\psi_c\\ &+ \psi_a(x_2)\,\psi_b(x_3)\,\psi_c(x_1) + \psi_a(x_3)\,\psi_b(x_1)\,\psi_c\end{aligned}$$

$$\rightarrow N = \frac{1}{\sqrt{6}}$$

(c) identical fermions: slater determinant:

$$\det\begin{pmatrix}\psi_a(x_1) & \psi_b(x_1) & \psi_c(x_1)\\ \psi_a(x_2) & \psi_b(x_2) & \psi_c(x_2)\\ \psi_a(x_3) & \psi_b(x_3) & \psi_c(x_3)\end{pmatrix}$$

remember that det is antisymmetric under exchange of 2 rows or columns. and vanishes for linear indep vectors

Consider a 3D infinite potential well

$$E = \frac{\hbar^2 \pi^2}{2ML^2} \left( n_x^2 + n_y^2 + n_z^2 \right)$$

$$n_x, n_y, n_z = 1, 2, 3, \ldots$$

suppose, we place 10 particles in this well, what is the lowest energy of this ten-particle state, when the particles are:

(a) identical spinless bosons

10 particles in the 111 - state

$E_{122} = E_{212} = E_{221}$ —— $= E_0 \cdot 9$

$E_{211} = E_{121} = E_{112}$ —— $= E_0 \cdot 6$

$E_{111}$ —— $= E_0 \cdot 3$

single particle levels

(b) identical spin 1 bosons

→ same as (a) → no Pauli exclusion

(c) identical fermions with spin $S = \frac{1}{2}$ note: single-particle energy does not depend on spin

$E_{221}$ ↑ $E_{122}$ ↑ $E_{212}$ —

$E_{211}$ ↑↓ $E_{121}$ ↑↓ $E_{112}$ ↑↓

$E_{111}$ ↑↓

(d) degeneracy of each level = $2s+1 \overset{s=\frac{3}{2}}{=} 4$