Relativistic Energy & Momentum

Required reading: Zwiebach §2.4

Suggested reading:
- French §5.1, §7.1-3
- Schwarz & Schwarz §4.1-4
- Hartle §5.2,3,5

Proper velocity:
This is largely a review of a problem on homework assignment 2.

- There is a way to define the velocity of a particle as a 4-vector
  \[ u^\mu = \frac{dx^\mu}{d\tau}, \]
  where the \( x^\mu \) are coordinates in some frame and \( \tau \) is the particle time along the particle’s worldline.
- Particles with timelike motion (moving at less than the speed of light) always have \( u^2 = u^\mu u_\mu = -1 \).
- You can similarly build an acceleration \( a^\mu = du^\mu/d\tau \), which is still a 4-vector. In the absence of forces, \( a^\mu = 0 \), as you’d expect.

Energy and momentum:

- It’s then reasonable to define a 4-vector for the momentum, in analogy with the nonrelativistic case, \( p^\mu = mu^\mu \), where \( m \) is the mass of the particle at rest (rest mass, which is the only kind of mass we’ll talk about).
- You can also figure out the components in terms of normal velocities \( d\vec{x}/dt \) from the homework.
- From our discussion of the proper velocity, it’s not hard to see that \( p^2 = -m^2 \). This is a general relation!
- What is the \( p^0 \) component? Well, look at a frame in which the particle velocity is nonrelativistic, so that \( d\tau \approx dt \) and therefore \( |\vec{u}|^2 \ll 1 \). Then
  \[ p^0 = m\sqrt{1 + |\vec{u}|^2} \approx m + \frac{1}{2}m|\vec{u}|^2 + \cdots. \]
  Notice that this is a rest mass plus kinetic energy, so it’s very reasonable that \( p^0 \) is the energy.
- Also on the homework, you’ll see that this relativistic energy makes good sense in terms of quantum mechanics.

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Massless particles:

- In quantum mechanics, we think of light as being quantized in terms of particles called photons.
- Since light rays travel in light-like directions, the proper time interval along them vanishes. That means we need a new definition for proper velocity and momentum.
- Parameterize the curve of the light ray by some arbitrary parameter $\lambda$; then, we can define $u^\mu = dx^\mu/d\lambda$. An affine parameter is one for which the acceleration $du^\mu/d\lambda = 0$, but there is not a unique affine parameter.
- The momentum is proportional to $u^\mu$, as for a massive particle, but the proportionality depends on the parameterization used.
- Massless particles have $p^2 = 0$, as generalizes from the massive case $p^\mu = [\hat{p}, \hat{p}]$.
- In terms of frequency and wave-vector, $p^\mu = [\omega, \hat{k}]$.

Conservation laws:

In Newtonian mechanics, we’re used to conserving energy, momentum, and mass. However, in relativity, it only really makes sense to have a conservation law for a covariant quantity, like a 4-vector, because then conservation works in all reference frames. So our conservation law is for the entire energy-momentum 4-vector $p^\mu$ in all reference frames.

Hints for relativistic mechanics problems:

- Always try to find a good inertial reference frame to use. In a collision between two particles, choose the center-of-mass frame in which the total momentum vanishes, for example.
- Work with covariant quantities and equations whenever you can!