Please answer all the following, to be returned in class Thursday, Feb. 23.

Final project information: If you intend to participate in the final project, you must fill out the “survey” on the last page of this homework and return it to me (either through campus mail, the envelope on my door, or in class) by Thursday, Feb. 23.

1. Evaluating divergent sums. Now you get to justify to yourself the $\zeta$-function method of evaluating the divergent sum in the zero-point energy.

(a) [10 points] Evaluate the sum with a smooth cut-off

$$ I(\epsilon) = \sum_{n=1}^{\infty} ne^{-\epsilon n} $$

and write out the singular and constant terms of the Laurent expansion of $I$ around $\epsilon = 0$. It is a principle of field theory that we have to impose cut-offs when evaluating seemingly divergent sums and integrals; these parameterize our ignorance of physics at high energies. The terms that diverge as the cut-offs vanish (or go to infinity, depending on the scheme) are canceled by additional infinite counterterms in the action. In the end, physics must be independent of the cut-off. Hence, we take the sum to be the finite constant in the Laurent expansion.

(b) [10 points] Here’s a more formal, mathematical derivation that relies on analytic continuation in a disguised way. This time, define

$$ I(x) = \sum_{n=1}^{\infty} (n + x) $$

If we just cancel terms in the sums blithely (here’s where the analytic continuation comes in), we find

$$ I(x + 1) - I(x) = -1 - x. $$

First, show that this difference relation is satisfied by

$$ I(x) = -\frac{1}{2} \left( x^2 + x \right) + I(0). $$

Next show from the definition of $I(x)$ that

$$ I(0) = -\sum_{n=1}^{\infty} (2n - 1), \quad I \left( -\frac{1}{2} \right) = \frac{1}{2} \sum_{n=1}^{\infty} (2n - 1). $$

Then put it all together to get $I(0) = -1/12$. 
2. Leading Regge trajectory. From equation (12.157) in the text, the center of mass angular momentum in the $x^2, x^3$ plane is

$$S \equiv M^{23} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_n^2 \alpha_n^3 - \alpha_n^3 \alpha_n^2 \right).$$

(You can check for yourself that this is Hermitian and normal-ordered, by which we mean it always vanishes on a ground state $|p^+, p^-\rangle$.) Consider a string with excitations only in the $x^2, x^3$ directions. For convenience, define

$$\alpha_n = \frac{1}{\sqrt{2}} \left( \alpha_n^2 + i \alpha_n^3 \right), \quad \bar{\alpha}_n = \frac{1}{\sqrt{2}} \left( \alpha_n^2 - i \alpha_n^3 \right),$$

so $\bar{\alpha}_{-n} = (\alpha_n)\dagger$. Also, throughout the rest of the problem, ignore oscillators in the other directions (don’t even bother writing them down).

(a) [10 points] Find the commutators of the $\alpha_n$ and $\bar{\alpha}_n$. Also, write $S$ and the mass-squared $m^2$ in terms of the $\alpha_n, \bar{\alpha}_n$.

(b) [10 points] Show that the maximum eigenvalue of $S$ is $1 + \alpha' m^2$ for any string. Further, argue that this maximum occurs for string states $|1\rangle = |p^+ \rangle |p^- \rangle$.

The relation $S \leq 1 + \alpha' m^2$ is why $\alpha'$ is sometimes known as the Regge slope; besides strings, mesons also follow this pattern, which is one reason string theory was originally used to describe the strong interaction. States that saturate the inequality are said to be on the leading Regge trajectory.

(c) [15 points] On a previous homework, you saw that strings with $\alpha_{-1}$ excitations describe rigidly rotating strings. Consider a leading Regge trajectory string with $N$ excitations of $\alpha_{-1}$. Determine the length of the string by taking the expectation value of $(\Delta X^2)^2 + (\Delta X^3)^2$ in the string state, where $\Delta X(\tau) = X(\tau, \pi) - X(\tau, 0)$. How is the energy/mass related to the length of the string? How does that relation compare to the classical relation $\ell = 2E/\pi T_0 = 4\alpha' m$ (see eqn (7.66) or problem 9.3 of the text)? Hint: It will probably help to define a $\Delta X$ and $\Delta X$, like $\alpha_n$ and $\bar{\alpha}_n$.

(d) [15 points] Consider a leading Regge trajectory string of large spin $S \gg 1$. Show that it can decay into 2 strings only by emitting a photon circularly polarized in the $x^2, x^3$ plane and becoming a string of spin $S - 1$. Find the energy of the photon. Work in the initial rest frame (center of momentum frame) in this part. Because the allowed decays are so limited, leading Regge trajectory strings decay very slowly compared to other highly excited strings. Hint: Call the 2 final strings...
1.2. Use angular momentum conservation and the Regge inequality above to show that $N \leq N_1 + N_2$, where $N = \alpha' m^2 + 1$ for any string. Then use conservation of energy to argue that $m \leq m_1 + m_2$. You should find that the only solution to these inequalities are $N_1 = N - 1$ with $N_2 = 1$ or vice versa.

3. **Zwiebach problem 12.10.** In this problem, you will find out the difference between oriented and unoriented strings.

(a) [5 points] As in the text. A clearly-labeled sketch is sufficient; you may denote the orientation of a string by an arrow along the curve of the string pointing in the direction of increasing $\sigma$.

(b) [10 points] As in the text.

(c) [5 points] As in the text.

(d) [10 points] As in the text. Note that $N^\perp = \sum_{n=1}^{\infty} \alpha^i_n \alpha^i_n = \alpha' m^2 + 1$.

(e) [10 points] As in the text.

**110 total points**
Final Project Questionnaire

Please fill out this form if you wish to participate in the final project. You should return these to me (Andrew) by Thursday, Feb. 23, either in class, to my mail box (Mail Code 452-48), or to the “Course Feedback” envelope on my office door (Downs 421). If you do not return this form, I will assume you are not participating. Also, if you are taking part in the final project, I’ll give out assignments Thursday, March 2, in class. I’ll also post them on the webpage, but you should probably come to the class to plan things with your partners.

The final project will consist of a question or series of questions of a bit higher difficulty than on a homework (after all, it is worth twice the points of a homework set). You will work in groups of 3 or 4 students (depending on the number of people participating). Each group will write a paper on their subject and, if we can find a time to meet, a short presentation during exam week. The final project is worth 200 points.

1. Name

2. Please rank order your top three choices of subject for your final project (1 for your favorite, then 2, then 3):
   A. D-brane actions and more on D-branes
   B. String thermodynamics vs black-hole thermodynamics
   C. Heterotic superstrings and connections to other strings
   D. M-theory (or strings meet 11 dimensions)
   E. More modern compactifications
   F. Supersymmetry and supergravity
   I’ll assign groups to put you with the project you want.

3. Please check off any times that you could attend a section to make group presentations.
   A. Mon., March 13, 2-4PM
   B. Weds., March 15, 2-4PM
   C. Thurs., March 16, 2-4PM
   D. Fri., March 17, 2-4PM
   If there is a time good for everyone, then I’ll arrange for us to meet in Lauritsen 469, the theory group seminar room.