Particle Quantization in Light-Cone Gauge

Required reading: Zwiebach §11.1,3

Suggested reading:
- Zwiebach §11.2
- Polchinski §1.2,3 (alternate action)
- A favorite quantum text

Light-cone gauge for the point particle:

Let’s remind ourselves of the worldline point particle action. It is just (minus) the mass times the proper time along the worldline, which we can write as

$$S = -m \int d\xi \sqrt{-\eta_{\mu\nu} \partial\xi X^\mu \partial\xi X^\nu} \equiv -m \int d\xi \sqrt{-(\partial\xi)^2}$$

for arbitrary worldline parameter $\xi$ (in Minkowski spacetime).

- By differentiating the action, the canonical momentum and equation of motion are
  $$p_\mu = \frac{m\partial\xi X^\mu}{\sqrt{-(\partial\xi)^2}} \;, \quad \partial\xi p_\mu = 0 \;.$$
  Note that we didn’t need to choose a gauge to find these equations.

- The action is invariant under change of coordinates $X$ in spacetime, so we are free to use light-cone coordinates. Let’s take a gauge
  $$X^+(\xi) = \frac{p^+}{m^2} \xi \;,$$
  much like we did for the string.

- Since we’ve lost a degree of freedom ($X^+$), there must be a constraint. If we look at the canonical momentum $p^+$, we find consistency only if
  $$(\partial\xi)^2 = -\frac{1}{m^2} \;.$$

- Then $p^\mu = m^2 \partial\xi X^\mu$, and the equation of motion becomes $\partial^2 X^\mu = 0$.

- Note that $p^-$ isn’t dynamical now, either; it is fixed by the constraint, which is also $p^2 = -m^2$:
  $$p^- = \frac{1}{2p^+} \left(p^ip^i + m^2\right) \;.$$
Further, solving the equations of motion requires

\[ X^- (\xi) = x^-_0 + \frac{p^-}{m^2} \xi, \quad X^i (\xi) = x^i_0 + \frac{p^i}{m^2} \xi. \]

Since \( p^- \) is determined by the constraint and \( X^+ \) by the gauge condition, our dynamical variables (degrees of freedom) are

\[ X^i, \quad x^0, \quad p^i, \quad p^+. \]

(By a redefinition, we could have chosen \( x^0 \) instead of \( X^0 \).)

**Schrödinger vs Heisenberg:**

Before we move onto quantization, let’s very briefly review two formalisms of quantum mechanics.

- **Schrödinger picture:** This is probably most familiar. In it, operators are typically time-independent (though there are some with explicit time dependence). Other than explicit time-dependence, time evolution occurs in states through the Schrödinger equation:

\[
i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle \Rightarrow |\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle.
\]

- **Heisenberg picture:** Here, we take the states to be time-independent (that is, they are always equal to the time zero state) but allow both explicit time dependence and implicit time dependence. The implicit time dependence arises through time evolution from the Hamiltonian. The time evolution of an operator \( \alpha \) is

\[
i \frac{d\alpha}{dt} = i \frac{\partial \alpha}{\partial t} + [\alpha(t), H] \Rightarrow \alpha(t) = e^{iHt} \alpha(0) e^{-iHt},
\]

where the partial derivative indicates explicit time dependence and the second equation is valid for an operator without explicit time dependence. Note that the above means that a Hamiltonian has no time dependence if it has no explicit time dependence.

- The commutator of two operators is the same in either formalism.
- I’ll refer you to the text and your favorite quantum textbook for more details.

**Quantizing the particle:**

This is just the first quantization of the particle, meaning the quantization of the motion of a single particle.
• The first step is to promote our degrees of freedom to operators

\[ X^i, \quad x_0, \quad p^i, \quad p^+ . \]

Then we can impose the commutators

\[ [x^i, p^j] = i\delta^{ij}, \quad [x_0^-, p^+] = ig^{-+} = -i , \]

as follows from the classical Poisson brackets. All other commutators vanish. Note that these commutators match the covariant form \([x^\mu, p^\nu] = ig^{\mu\nu}\) specified to light-cone coordinates.

• From the classical constraints, we have to define other operators

\[ X^+ \equiv \frac{p^+}{m^2} \xi, \quad X^- \equiv x_0^- + \frac{p^+}{m^2} \xi, \quad p^- \equiv \frac{1}{2p^+} (p^j p^i + m^2) . \]

• Note that the commutators for these “derived” operators are not covariant; in particular, \([X^+, p^-] = 0\).

• We need to be a bit clever to come up with the Hamiltonian here, since we’re working with a constrained system and don’t want to go through the full formalism for constraints. Let’s remember, though, that \(p^- = i\partial_+\). Then

\[ i\partial_\xi = \frac{p^+}{m^2} p^- \Rightarrow H = \frac{p^+ p^-}{m^2} = \frac{1}{2m^2} (p^j p^i + m^2) \]

for Schrödinger’s equation to work.

• With some commutator algebra, you can reproduce the classical equations for the evolution of the operators (for example, \(\partial_\xi x^i = p^i/m^2\)) in the Heisenberg picture.

• Because the \(X\) and \(p\) operators do not commute, we can’t label states by their eigenvalues under both sets. Since momenta commute with the Hamiltonian, let’s use those. Therefore, we can build a basis of states \(|p^+, p^i\rangle\).

• A general state in the Schrödinger picture is a superposition (time-dependent!) of the basis states:

\[ |\Psi(\xi)\rangle = \int dp^+ dp^{i-1} p^i \psi(\xi; p^+, p^i) |p^+, p^i\rangle . \]

The Schrödinger equation then becomes

\[ i\partial_\xi \psi(\xi; p^+, p^i) = \frac{1}{2m^2} (p^j p^i + m^2) \psi(\xi; p^+, p^i) . \]