Lorentz Transformations and Special Relativity

Required reading: Zwiebach §2.1,2,6

Suggested reading:
- French §3.7-10, §4.1-5, §5.1 (a little less technical)
- Schwarz & Schwarz §1.2-6, §3.1-4 (more mathematical)

Units:
Units are quantities used for reference in making measurements; when we say that something is 2 kg, we mean that its mass is twice (to within significant digits) the mass of the standard kilogram locked up in France. We are perfectly free to choose other units than SI, though; cgs units are popular in astrophysics, for example.
We will use natural units, so named because they are based on fundamental quantities of the universe.

- We set the speed of light $c = 1$. Relativistically, $c$ is a fundamental speed, so this makes some sense. What we are saying is that, in our units, length and time are the same; normally $c$ would be a conversion factor between units of length and time (note that the equation for the position of a light pulse moving along the $x$ axis is $x = ct$).

- When we get into quantum mechanics, it will also be natural to set $\hbar = 1$. From the commutator $[p, x] = -i\hbar$, we see that, in our units, momentum and length have opposite units $[p] = 1/[L]$.

- If we ever do thermodynamics (which probably won’t come up), we’ll set Boltzmann’s constant $k = 1$. Because the average thermal energy of an ideal gas molecule (for example) is $E = 3kT/2$, temperature and energy have the same units.

- It might also seem natural to set Newton’s constant $G$ to unity. Making that choice as well as the above gives Planck units, and all quantities are dimensionless in those units. However, we will not use Planck units!
Setting $G = 1$ is equivalent to setting a gravitational length scale to one; in string theory, there is a different length scale $\ell_s$ corresponding to a typical size of quantum strings. For our purposes, we will not set either length scale to unity.

Postulates of relativity:
There are two basic postulates for relativity:

- All inertial frames are equivalent (this is already true in Newtonian mechanics).
The speed of light in a vacuum, \( c \), is the same in all inertial frames (this is why we set \( c = 1 \)).

**A boost:**

Perhaps surprisingly, we can in fact derive the relativistic rule for transforming one inertial frame to another from these postulates (we follow French here). So we start by imagining that we have 2 frames, \( S \) and \( S' \), which moves with speed \( v \) along the positive \( x \) axis relative to the \( S \) frame. In the \( S \) frame, we have coordinates \( x, y, z, t \), and \( x', y', z', t' \) in the \( S' \) frame.

- The transformation from one frame to another must be linear. Otherwise, motion with constant velocity in frame \( S \) would look like accelerated motion in \( S' \). Therefore, we can write

  \[
  x = ax' + bt', \quad x' = ax - bt',
  \]

invoking the fact that the transformation \( S \to S' \) is different than \( S' \to S \) only in the change of sign of the relative velocity.

- Since the origin \( x' = 0 \) moves with velocity \( v \) in the \( S \) frame, its position is \( x = vt \). This leads to \( b/a = v \) in the above.

- Now, suppose a flash of light is emitted from the origin at the time when \( x = 0 \) and \( x' = 0 \) are aligned \((t = t' = 0)\). Because of the 2nd postulate, the light front follows the lines \( x = t \) and \( x' = t' \). Then (1) are only consistent if \( a = 1/\sqrt{1-v^2} \). Normally, we will denote this coefficient \( \gamma = 1/\sqrt{1-v^2} \).

- We can also solve (1) for \( t \) in terms of \( x', t' \) and \( t' \) in terms of \( x, t \).

- Because of the equivalence of frames, we can also conclude that \( y = y' \), \( z = z' \).

In the end, we can write this *Lorentz boost* as

\[
\begin{bmatrix}
  t' \\
x' \\
y' \\
z'
\end{bmatrix}
= \begin{bmatrix}
  \gamma & -\gamma v & 0 & 0 \\
-\gamma v & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
t \\
x \\
y \\
z
\end{bmatrix}.
\]

You can similarly write out boosts in the \( y \) and \( z \) directions.

**4-vectors:**

We’ll now introduce some notation. Let’s write the 4D vector \([t, x, y, z]\) as \( x^\mu \), where \( x^0 = t \) and \( \mu = 1, 2, 3 \) give \( x, y, z \) respectively. We will always use Greek letters to indicate all of spacetime and roman letters to represent just space coordinates. So, for example, \( \mu = 0 \cdots 3 \), but \( i = 1 \cdots 3 \). A vector symbol
will also denote just the spatial part of a 4-vector. Then we can write also 
\( x'^\mu = [t', x', y', z'] \). In this notation, a matrix should have two indices, so we 
write the Lorentz transformation as 
\[ x'^\mu = L^\mu_{\nu} x^\nu . \]

We have introduced Einstein summation conventions, in which a lowered index 
contracted with an identical raised index is summed over. In the above equation, 
there is implicitly \( \sum_0^3 \). As an exercise, work out for yourself that the 
summation formula above does the correct matrix multiplication. Note that the 
individual components \( L^\mu_{\nu} \) are just numbers and can be commuted around.

**The metric:**

- It’s not too hard to show that the quantity 
  \[ ds^2 \equiv -d\tau^2 \equiv -(dx^0)^2 + |d\vec{x}|^2 \]
  is left unchanged by the boost we’ve given above. \( ds^2 \) is called the line 
element or alternately the metric. For a positive \( ds^2 \), \( ds \) gives the proper 
distance of two points spacelike separated by \( dx^\mu \). Meanwhile, \( d\tau \) is called 
the proper time between two points when \( d\tau^2 \) is positive (the points are 
timelike separated. Note: our \( d\tau^2 \) is the same as Zwiebach’s \( ds^2 \), so there 
is a sign difference.

- We can also define a matrix \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \), which we will also call 
the metric. Then using summation conventions, 
  \[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu . \]

- We can lower an index on a 4-vector by contracting with the metric: 
  \[ a_\mu = \eta_{\mu\nu} a^\nu . \]

- The matrix inverse of \( \eta_{\mu\nu} \) is denoted with raised indices and is just \( \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). The inverse property is written as 
  \[ \eta^{\mu\lambda} \eta_{\lambda\nu} = \delta^\mu_{\nu} \]
  using the Kronecker delta symbol.

- We can raise indices with the inverse metric: \( a^\mu = \eta^{\mu\nu} a_\nu \).

- For two 4-vectors \( a^\mu, b^\mu \), it’s not too hard to show that the scalar product 
  \[ a \cdot b = a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu \]
  is invariant under the boost written above.
**Lorentz transformations:**

What are all the transformations that leave the scalar product (3) (and therefore the metric (2)) invariant? Let’s examine the scalar product:

\[ a' \cdot b' = (L^\mu_\lambda a^\lambda) \eta_{\mu\nu} (L^\nu_\rho b^\rho) . \]

Clearly, this works for all matrices \( L \) such that

\[ L^\mu_\lambda L^\nu_\rho \eta_{\mu\nu} = \eta_{\lambda\rho} . \]

(Show yourself that this is \( L^T \eta L \) in matrix notation.) Any transformation \( L^\mu_\nu \) that leaves the metric invariant as above is a Lorentz transformation. These include all the boosts, spatial rotations, and mixtures of them. Reflections on coordinate axes also are Lorentz transformations. It is possible to see that all Lorentz transformations have \( \det L = \pm 1 \).

- **Lorentz transformations** with \( \det L = 1 \) are *proper* and include all boosts and rotations. These form a group, called \( SO(3, 1) \) in 4 dimensions.
- **Improper** transformations have negative determinant. Taking all Lorentz transformations, proper and improper, gives the group \( O(3, 1) \).

**Minkowski diagrams and worldlines:**

Consider making a graph of spacetime, as below. Starting at the origin, we can trace out curves corresponding to light rays \( (x = \pm t) \). Note that these lines have a slope of 45°. We can then also follow the path of a particle in spacetime, \( x(t) \), on the graph. This particle worldline can never have a slope less than 45°, since the particle can never travel greater than the speed of light!

The left figure shows the (dashed) worldline of a particle traveling between two light rays (diagonals). The right figure shows the lightcone in 3D (ie, suppressing the \( z \) axis) and labels the timelike and spacelike separated regions.
Tensors and covariance:

We’ve already seen how spacetime (4-)vectors change under Lorentz transformations. Now let’s discuss some other objects:

- Denote the matrix inverse of the Lorentz transformation $L^\mu_\nu$, by $L^\mu_\nu$, so that $L^\mu_\lambda L^\lambda_\nu = \delta^\mu_\nu$, etc.
- A vector with its index lowered: $a_\mu = a^\nu \eta_{\mu\nu}$. Show for yourself that $a'^\mu = L^\mu_\nu a^\nu \Rightarrow a'_\mu = L^\mu_\nu a_\nu$.
- A tensor is an object that transforms like a product of vectors (some with upper indices and some with lower). Each index of the tensor transforms with its own copy of $L$: $T^\rho^\mu^\nu^\lambda_{\alpha\beta} = L^\rho_\mu L^\nu_\sigma L^\lambda_\gamma L^\sigma_\delta T^\rho^\mu^\nu^\lambda_{\gamma\delta}$, for example.
- The metric $\eta$ is a tensor (with upper or lower indices). You’ll get to show this to yourself on the homework. Another important tensor we’ll talk about in class will be the electromagnetic field strength tensor.
- You can convince yourself that two contracted indices do not transform (that is, $a \cdot b$ is invariant, and $T^\mu_\nu$ transforms like a vector with only the $\nu$ index).

A covariant equation is one in which the free (uncontracted) indices match on the LHS and RHS. Since both sides of the equation transform in the same way, they are valid in any inertial frame.

Extra dimensions:

Suppose that instead of 4D (3 space and 1 time), we have $D = d + 1$ dimensions. Then

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad ds^2 = -dt^2 + (dx^1)^2 + \cdots + (dx^d)^2.$$

As in 4D, the Lorentz transformations, the Lorentz transformations are those $L^\mu_\nu$ that satisfy $L^\mu_\lambda L^\lambda_\rho \eta_{\mu\nu} = \eta_{\lambda\rho}$.

- They include boosts along the $d$ spatial dimensions and rotations of each pair of spatial dimensions. So with $d = 3$ there are $3C_2 = 3$ (“3 choose 2”) planes in which to rotate, $(x, y)$, $(y, z)$, and $(z, x)$, but there are more rotations in higher $d$. For example, there are $5C_2 = 10$ rotations in 5 spatial dimensions.
- The group of proper transformations is now $SO(d, 1)$, which includes the usual 4D Lorentz transformations.

- If one of the dimensions is a circle, rather than a full real line, there is still a local Lorentz transformation; that is, the metric is still valid for small distances. However, when you travel around the whole circle, clearly the actual proper distance is zero rather than $2\pi R$ as given by the metric. So only the noncompact dimensions have the full structure of Lorentz invariance. And if $dx = 0$ along the circle direction, we do that the metric for the smaller-dimensional spacetime. More on this later.