Open string operator quantization

Required reading: Zwiebach §12.1-4

Suggested reading:  
Polchinski §1.3  
Green, Schwarz, & Witten §2.3.1 (upto eq. (2.3.13))

The light-cone string as a field theory:

Today we will discuss the quantization of an open string; we will just be considering first quantization, because we know the action for a single string. (Incidentally, it isn’t known in general how to carry out second quantization for strings, and it’s a pretty technical research subject.) However, to do first quantization of strings, we have to do second quantization on the worldsheet, since the light-cone action is a field theory!

- For now, let’s work with all Neumann boundary conditions.
- The light-cone gauge action for an open string is
  \[ S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left( -4\alpha' p^+ \partial_\tau X^- + \left( \partial_\tau X^i \right)^2 - \left( \partial_\sigma X^i \right)^2 \right). \]
- Recall that \( X^- \) is not really a degree of freedom, since it is fixed by the constraints, and it won’t enter into the Hamiltonian due to the form of its action.
- Each of the remaining transverse coordinates \( X^i \) has the action
  \[ S = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \partial_\alpha X \partial^\alpha X. \]
  This is just the action for a massless field in two dimensions, so we can just combine the results we have for scalar field theory with what we know about the mode expansion of \( X \).
- For an open string, we had mode expansion
  \[ X^i(\tau, \sigma) = x^i_0 + \sqrt{2}\alpha' \alpha^i_0 \tau + i\sqrt{2}\alpha' \sum_{n \neq 0} \frac{\alpha^i_n}{n} \cos(n\sigma) e^{-in\tau}. \]
  Let’s compare to the similar expansion for a scalar field in 2-dimensions
  \[ \phi(x) = \sum_p \frac{1}{\sqrt{2LE_p}} \left( a_p e^{-iE_p t} e^{ipx} + a_p^* e^{iE_p t} e^{ipx} \right). \]
  - To compare, we should really identify \( t = \tau, \ x = \sigma, \ L = \pi, \) and \( \phi = X/\sqrt{2\pi\alpha'} \).
– Apparently, there is a slight difference due to the Neumann boundary conditions on the string vs the periodic bc for the field. This isn’t consequential.

– Also, because the field $X$ for the string is massless from the worldsheet point of view, the zero momentum part looks different.

– It’s pretty clear, though, that $\alpha_n^i$ is an annihilation operator on the worldsheet for $n > 0$. Also, $\alpha_{-n}^i = (\alpha_n^i)\dagger$ for $n > 0$. We can set $\alpha_n = \sqrt{n}\alpha_n$.

– For a massless $\phi$ and nonzero momentum, we require $p = n$ for $n \in 2\mathbb{Z}$. That means the energy is $\sqrt{n}$. We can then relate the commutators

$$[a_n, a_m^\dagger] = \delta_{n,m} \Rightarrow \left[\alpha_n^i, \alpha_m^j\right] = n\delta_{n,m}\delta^{ij} \quad (n > 0).$$

For the string, though, $n$ can be any integer.

- To understand the zero-momentum part, we can’t just use the field theory analogy. However, if we set the zero momentum piece to $q^i(\tau)$, we find that the action for $q^i$ is just

$$S_q = \frac{1}{4\alpha'} \int d\tau (q^i)^2,$$

which is just that for a free particle. Since the Heisenberg equation says that $q$ should be linear in time, we find

$$q^i(\tau) = x^i_0 + 2\alpha' p^i \tau, \quad [x^i_0, p^i] = i\delta^{ij},$$

including the standard commutator. We also find $\alpha_0^i = \sqrt{2\alpha'} p^i$.

- You can also start with the (equal-time) commutator between each $X^i$ and its canonical momentum $P^i$

$$[X(\tau, \sigma), P(\tau, \sigma')] = i\delta(\sigma - \sigma')$$

to work out the oscillator commutators. The text describes this approach.

**Hamiltonian:**

Let’s move on to discuss the Hamiltonian.

- From the action above, we can write the Hamiltonian as

$$H = \int_0^\pi d\sigma \left( \pi\alpha' P^i P^i + \frac{1}{4\pi\alpha'} \partial_\sigma X^i \partial_\sigma X^i \right), \quad P^i = \frac{1}{2\pi\alpha'} \partial_\tau X^i.$$
For the zero-modes, the Hamiltonian just comes out to be \( \alpha' p^i p^j \), the center of mass momentum-squared for the string (in the transverse directions). From our results for scalar fields, then, we find

\[
H = \alpha' p^i p^j + \sum_{n=1}^{\infty} a^i_n a^i_n + C = \alpha' p^i p^j + \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + C.
\]

We’ve combined all the zero point energies into one term.

Let’s remember that we defined the \( X^- \) oscillators in terms of Virasoro modes (now Virasoro operators):

\[
\sqrt{2\alpha' \alpha_n^-} = \frac{1}{p^+} L^+_n = \frac{1}{p^+} \left( \frac{1}{2} \sum_p \alpha^i_{n-p} \alpha^i_p \right).
\]

In particular,

\[
L^+_0 = \frac{1}{2} \left( \alpha^i_0 \alpha^i_0 + \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{n=1}^{\infty} \alpha^i_n \alpha^i_{-n} \right)
\]

\[
= \frac{1}{2} \alpha^i_0 \alpha^i_0 + \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \frac{1}{2} \sum_{n=1}^{\infty} [\alpha^i_n, \alpha^i_{-n}]
\]

\[
= \alpha' p^i p^j + \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \frac{D-2}{2} \sum_{n=1}^{\infty} n.
\]

The last term is an infinite-looking zero-point term. As long as we identify this constant with the constant \( C \), we have \( H = L^+_0 \).

From the relation \( \alpha_0^- = \sqrt{2\alpha' p^-} \), we have

\[
H = L^+_0 = 2\alpha' p^+ p^-.
\]

If we remember from the light-cone gauge point particle, this is appropriate for a Hamiltonian that satisfies the Schrödinger equation \( i \partial_x = H \).

The mass spectrum is now modified from the classical formula! We have here

\[
m^2 = 2p^+ p^- - p^i p^j = \frac{1}{\alpha'} \left( \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + C \right).
\]

Due to the normalization of the \( \alpha^i_n \), each excitation at level \( n \) adds \( n/\alpha' \) to the mass-squared. Note that the zero-point energy has real meaning here, as it contributes to the mass of the string state! (That fact is related to the fact that the worldsheet theory includes gravity before we gauge-fix.)
Virasoro operator algebra:

Since it’s fairly important if you progress in string theory, let’s briefly discuss the commutator algebra of the Virasoro operators.

- First note that the conjugate operators are \((L^+_n)^\dagger = L^-_{-n}\).
- We can make use of the commutator relation \([AB, C] = A[B, C] + [A, C]B\) to find
  \[
  [L^+_m, \alpha^i_n] = -n\alpha^i_{m+n}, \quad [L^+_m, x^i_0] = -i\sqrt{2}\alpha^i_m .
  \]
- Iterating, we find
  \[
  [L^+_m, L^+_n] = \frac{1}{2} \sum_{p=\infty}^{\infty} \left( (p-n)\alpha^i_{m+n-p}\alpha^i_p - p\alpha^i_{n-p}\alpha^i_{m+p} \right) .
  \]

If \(n = -m\), we have nontrivial commutators; in the end we find
\[
[L^+_m, L^+_n] = (m-n) L^+_m + A_m \delta_{m,-n}
\]
with
\[
A_m = (D - 2) \sum_{p=1}^{\infty} p^2 - \frac{D - 2}{2} \sum_{p=1}^{m} p(p-m) = D + \frac{D - 2}{12} (m^3 - m) .
\]

Getting this last requires figuring out which terms have nonzero commutators and distributing the sums (a dicey proposition with divergent sums, but it works ok here). \(D\) is formally infinite, and we’ll discuss it below.

- Let’s look at the action of the Virasoro operators on the string coordinates. With some algebra, we find
  \[
  [L^+_m, X^i(\tau, \sigma)] = -i\sqrt{2}\alpha^i_m X^i(\tau, \sigma) + e^{im\tau}\sin(m\sigma) \partial_{\sigma} X^i - e^{im\tau}\cos(m\sigma) \partial_{\tau} X^i .
  \]
  This second line shows that \(L^+_m\) generates a change of coordinates
  \[
  (\tau, \sigma) \rightarrow (\tau, \sigma) + e^{im\tau}(-i\cos(m\sigma), \sin(m\sigma)) .
  \]

Zeta functions:

Here we will use the magic of analytical continuation, and you will justify the formulae below on the homework. First note that the Riemann zeta function is defined as
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \geq 2 .
\]
Fortunately, \(\zeta(s)\) has a unique analytic continuation for all complex \(s \neq 1\), so we can evaluate what we want.
• The first sum we need corresponds to $\zeta(-1) = -1/12$ (odd that a sum of positive integers is a negative fraction, isn’t it?). We find that the normal ordering constant in $L^+_0 = H$ is

$$C = -\frac{D - 2}{24}.$$ 

• The other infinite sum we have is $\zeta(-2) = 0$. That means that $D = 0$ in $A_m$. 