Quantization of a Scalar Field

Required reading: Zwiebach §10.1-4, §11.4

Suggested reading:
- Your favorite quantum text
- Any quantum field theory text

Quantizing a harmonic oscillator:

Let’s start by reviewing a harmonic oscillator and its quantization.

- The action for a harmonic oscillator with coordinate $q$ (of general dimension) is
  \[ S = A \int dt \left( \frac{1}{2} \dot{q}^2 - \frac{\omega^2}{2} q^2 \right). \]
  Here, $A$ is a coefficient to correct the dimensions (which plays the same role as the mass), and $\omega$ is a frequency.

- The Hamiltonian is simply
  \[ H = \frac{p^2}{2A} + \frac{A\omega^2}{2} q^2, \quad p = A\dot{q}. \]

- We can also rewrite the Hamiltonian in terms of the variable
  \[ a(t) = \left( \frac{A\omega}{2} \right)^{1/2} q(t) + i \left( \frac{1}{2A\omega} \right)^{1/2} p(t) \]
  and its complex conjugate, giving
  \[ H = \omega a^\dagger a + C. \]
  We’ve left the zero-point energy $C$, which arises due to ordering ambiguities, as undetermined for now.

- In quantum mechanics,
  \[ [q, p] = i \Rightarrow [a, a^\dagger] = 1, \]
  as you can check yourself.

- The lowest energy state $|0\rangle$ has the property that $a|0\rangle = 0$. To build other states, we can act with $a^\dagger$ repeatedly. The properly normalized $n$th excited state is $(1/\sqrt{n!})(a^\dagger)^n|0\rangle$, and it has energy $n\omega + C$. Hence we call $a^\dagger$ a raising or creation operator. (Inversely, $a$ is a lowering or annihilation operator.)
The classical scalar field:

As mentioned in the text, a scalar field is simply a function of spacetime, $\phi(x^\mu)$ and is invariant under Lorentz (or coordinate) transformations. This means that a scalar $\phi$ is a zero-index tensor. All observers agree on the value of $\phi$ at a fixed point in spacetime, even if they disagree on what coordinates to call that point by.

- Let’s describe an action for a scalar field. There are many actions we could take, but let’s just use the simplest.

  - As a generalization of kinetic energy, the simplest form is a kinetic energy density that looks non-relativistic:
    $\quad T = \frac{1}{2} (\partial_0 \phi)^2$.
  
  - This kinetic energy density is not Lorentz invariant, and we’d like a Lorentz invariant action. So let’s include a potential energy density due to the spatial gradient of the field
    $\quad V_T = \frac{1}{2} |\vec{\partial}\phi|^2$.

    We call this potential $V_T$ to emphasize that we add it to make the kinetic energy Lorentz invariant.

  - Finally, we can add a general potential $V(\phi)$ (or even include derivatives of $\phi$ in a Lorentz invariant way). If $\phi = 0$ is a minimum of the potential, the simplest choice is
    $\quad V = \frac{1}{2} m^2 \phi^2$,

    where $m$ is a mass (to match the dimensionality of the other terms).

  - In the end, the action becomes
    $\quad S = \frac{1}{2} \int d^D x \left( -\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 \right)$.

- The canonical momentum density conjugate to $\phi$ comes from the kinetic part of the action:
  $\quad \Pi = \frac{\partial L}{\partial (\partial_0 \phi)} = \partial_0 \phi$.

The Hamiltonian is therefore
  $\quad H = \frac{1}{2} \int d^D x \left( \Pi^2 + |\vec{\partial}\phi|^2 + m^2 \phi^2 \right)$.

- The equation of motion is the *Klein-Gordon equation*
  $\quad (\partial^\mu \partial_\mu - m^2) \phi = \left( -\partial_t^2 + \vec{\partial}^2 - m^2 \right) \phi = 0$.
• To study solutions of the EOM, let’s take a Fourier transform

\[ \phi(x) = \int \frac{d^D p}{(2\pi)^D} \phi(p)e^{ip \cdot x}. \]

For reality of \( \phi(x) \), we need \( \phi^*(p) = \phi(-p) \). (You can work it out or read the text.) We now have the field as a function of momentum rather than position.

• The equation of motion becomes

\[ \int \frac{d^D p}{(2\pi)^D} (p^2 + m^2) \phi(p)e^{ip \cdot x} = 0, \]

which implies that \( \phi(p) \) vanishes off the mass-shell \( p^2 = -m^2 \). This is the same as the constraint for the light-cone point particle!

• Now consider that we use light-cone coordinates and Fourier transform on \( x^-, x^+ \). Then we have

\[ \phi(x) = \int \frac{dp^+}{2\pi} \frac{d^{d-1} p^i}{(2\pi)^{d-1}} \phi(x^+, p^+, p^i)e^{-ix^-p^+ + ix^i p^i}. \]

• Since the Laplacian in light-cone coordinates is

\[ \partial^2 = -2\partial_+ \partial_- + \partial_i \partial_i, \]

the equation of motion becomes

\[ \left( i\partial_+ - \frac{1}{2p^+} \left( p^i p^i + m^2 \right) \right) \phi(x^+, p^+, p^i) = 0. \]

This equation is actually the same as the Schrödinger equation for the light-cone point particle wavefunction!

**Quantizing the scalar field:**

We’ll roughly follow the text, but we’ll take a little bit of a more conventional approach. For simplicity, we’ll assume that space is a box of volume \( V \) (with each side length \( L \)), so the momentum has to satisfy

\[ \vec{p} L = 2\pi \vec{n}, \quad n^i \in \mathbb{Z} \]

for periodicity of the field.

• Notice that the scalar field action is something like the action for a harmonic oscillator with a frequency

\[ \omega^2 \sim |\vec{p}|^2 + m^2 \]

because \( \partial = i\vec{p} \) for a single momentum mode. This is, of course, just at the level of analogy at this point.
To further the analogy a bit, we can take a lowering operator for each momentum

$$a_p(t) = \frac{1}{\sqrt{V}} \int d^dx \left( \frac{E_p}{2} \phi(x) + i \frac{1}{\sqrt{2E_p}} \Pi(x) \right) e^{-ipx}.$$ 

We have defined the relativistic energy

$$E_p = \sqrt{p^2 + m^2}.$$ 

On the homework, you will show that

$$\delta^d(x - x') = \frac{1}{V} \sum_p e^{ip(x-x')}.$$ 

Using that result, we can show that our previous Hamiltonian is also written as

$$H = \sum_p \left( E_p a_p^\dagger a_p + C_p \right),$$ 

where again $C_p$ is a constant related to ordering ambiguities. In field theory, we can freely set this equal to zero.

We take the commutators of the creation and annihilation operators to be

$$[a_p, a_k^\dagger] = \delta_{p,k}$$

with other commutators vanishing. On the homework, you will use these to find the commutator of $\phi$ and $\Pi$.

Using the commutator and the Hamiltonian above, we can see that

$$a_p(t) = a_p e^{-iE_p t}, \quad a_p^\dagger(t) = a_p^\dagger e^{iE_p t},$$

where the “coefficient” operators are the constant Schrödinger operators.

Then we can write

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_p \frac{1}{\sqrt{2E_p}} \left( a_p e^{ipx} + a_p^\dagger e^{-ipx} \right).$$

The state space is just like that of a harmonic oscillator, except we imagine that the ground state $|\Omega\rangle$ is the vacuum of spacetime (that is, there are no particles). Then the excited states have particles

$$a^\dagger_p|\Omega\rangle \quad \text{one particle of momentum } p$$

$$a^\dagger_p a^\dagger_k |\Omega\rangle \quad \text{two particles of momenta } p, k$$

etc...
• Note that the one particle states are the same as those of a point particle, especially if we write the momentum in light-cone components:

\[ a_{p^+, p^i}^\dagger |\Omega\rangle \leftrightarrow |p^{\pm}, p^i\rangle . \]

We have second quantized the particle, meaning we know the quantum states of a single particle but can also have many particles. We also can create and destroy particles.