1 Problem 1

The Hamiltonian is

\[ H = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A}(x) \right)^2 + e\Phi(x) \]  

(1)

(a) The relevant equation is

\[ \dot{x}_i = \frac{\partial H}{\partial p_i} \]  

(2)

Performing the differentiation and rearranging gives

\[ \vec{p} = m\ddot{x} + \frac{e}{c} \vec{A}(x), \]  

(3)

which we recognise as the standard expression for the canonical momentum of a charge in electrodynamics.

For a particle at rest in a magnetic field, the momentum is just

\[ \vec{p} = \frac{e}{c} \vec{A}(x) \]  

(4)

and so, if the potential \( \Phi = 0 \),

\[ H = 0. \]  

(5)

This is just because the magnetic field does not affect a stationary charge.
(b) To calculate the force on the particle, we need the second set of Hamilton’s equations.

\[ \dot{p}_i = -\frac{\partial H}{\partial x_i} \]

\[ = -e \frac{\partial \Phi}{\partial x_i} + e \frac{\partial A_j}{c} \dot{x}_j. \]  

But, using our expression (3) for the canonical momentum we find that

\[ \dot{p} = \ddot{x} + \frac{e}{c} \frac{d \vec{A}(x)}{dt} \]  

It takes a little work to reduce these equations to the form of the Lorentz force law. It helps to notice that the cross product in the Lorentz force can be written in terms of the vector potential as follows:

\[ (\dot{x} \times \vec{B})_i = \epsilon_{ijk} \dot{x}_j \epsilon_{klm} \partial_l A_m \]

\[ = \dot{x}_j \partial_i A_j - \dot{x}_j \partial_j A_i. \]  

The first term in the last equality above is also present in our expression (7) so we just need the second term. Using the chain rule in (8) leads to

\[ \dot{p}_i = m \ddot{x}_i + \frac{e}{c} \left( \frac{\partial A_i}{\partial x_j} \dot{x}_j + \frac{\partial A_i}{\partial t} \right) \]  

Now we can eliminate \( \dot{p}_i \) between (7) and (11) and use (10) to find

\[ m \ddot{x}_i = -e \frac{\partial \Phi}{\partial x_i} - \frac{e}{c} \frac{\partial A_i}{\partial t} + \frac{e}{c} \epsilon_{ijk} \dot{x}_j B_k. \]  

Since

\[ E_i = -\frac{\partial \Phi}{\partial x_i} - \frac{1}{c} \frac{\partial A_i}{\partial t}, \]

this is the Lorentz force law.

(c) The Lagrangian is given in terms of the Hamiltonian by

\[ L(q, \dot{q}) = \dot{q} p - H(p, q) \]  

A little algebra shows that in this case the Lagrangian is

\[ L = \frac{1}{2} m \dot{x}^2 + \frac{e}{c} \dot{x} \cdot \vec{A} - e \Phi(x). \]
(d) There was a slight typo in the statement of the problem which could cause some trouble here. The vector potential for a constant magnetic field

\[ \vec{A} = \frac{1}{2} \vec{B} \times \vec{x} \]  

which differs by a sign to the potential in the problem.

In cylindrical coordinates with unit vectors \( \hat{r}, \hat{\theta} \) and \( \hat{z} \), the vector potential is \( \vec{A} = 1/2rB\hat{\theta} \). Therefore the Lagrangian is

\[ L = \frac{1}{2} m (\ddot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + \frac{e}{2c} r^2 B \dot{\theta}, \]  

since there is no electric field.

The equations of motion are given by the Euler-Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \]  

(18)

In this case, these equations become

\[ m\ddot{r} = m r \dot{\theta}^2 + \frac{e}{c} r B \dot{\theta} \]  

and

\[ \frac{d}{dt} \left( m r^2 \dot{\theta} + \frac{e}{2c} r^2 B \right) = 0 \]  

(20)

(there is no force in the \( z \) direction.)

These equations are the same as the Lorentz force equations. In fact, the Lorentz force law is

\[ m \left( \ddot{r} \hat{r} + 2i \dot{\theta} \hat{\theta} + r \ddot{\theta} \hat{\theta} - r \dot{\theta} \dot{\hat{r}} + \dot{z} \hat{z} \right) = \frac{e}{c} \left( -\dot{r} B \hat{\theta} + r \dot{\theta} B \hat{r} \right) \]  

(21)

which is the same as the equations we deduced from the Euler-Lagrange equations.

The Hamiltonian is

\[ H = \frac{1}{2m} (\vec{p} - \frac{e}{2c} r B \dot{\theta})^2 + e \Phi(r, \theta, z). \]  

(22)
2 Problem 2. (Zwiebach problem 4.3)

(a) The string displacement is
\[ y(t, x) = h_+(x - v_0 t) + h_-(x + v_0 t) \]  
(23)

We impose Dirichlet boundary conditions, so
\[ y(t, 0) = 0 = h_+(-v_0 t) + h_-(v_0 t) \]  
(24)

Hence,
\[ h_+(u) = -h_-(u). \]  
(25)

The boundary condition at the other end of the string is
\[ y(t, a) = 0 = h_+(a - v_0 t) + h_-(a + v_0 t). \]  
(26)

Hence,
\[ h_+(u) = h_+(u + 2a), \]  
(27)
as required.

(b) We are given initial conditions
\[ y(0, x) = 0 \]  
(28)
\[ \frac{\partial y(0, x)}{\partial t} = v_0 \frac{x}{a} \left(1 - \frac{x}{a}\right) \]  
(29)

The first of these conditions implies that \( h_-(x) = -h_+(x) \), so we just have to find the function \( h_+ \). The second condition, expressed in terms of \( h_+ \), is
\[ -2v_0 h'_+ = v_0 \frac{x}{a} \left(1 - \frac{x}{a}\right) \]  
(30)

This is an ordinary differential equation for the function \( h_+(x) \), and it applies for \( x \in (0, a) \), since that is the region in which the initial condition is given. The solution is
\[ h_+(x) = -\frac{x^2}{4a} + \frac{x^3}{6a^2} + c. \]  
(31)

This solution only applies for \( x \in (0, a) \). To find the solution for \( x \in (-a, 0) \), notice that
\[ h_+(x) = -h_-(x) = +h_+(-x) \]  
(32)
so that
\[ h_+(x) = -\frac{x^2}{4a} - \frac{x^3}{6a^2} + c \] (33)
for \( x \in (-a, 0) \). In these equations \( c \) is an arbitrary constant; we will see that it has no effect on the displacement \( y(t, x) \).

Now we know \( h_+(x) \) in the region \((-a, a)\); this is sufficient to determine \( h_+ \) everywhere since \( h_+(x) = h_+(x+2a) \) (that is, \( h_+(x) \) is a periodic function with period \( 2a \)).

(c) The domain \( D \) is the diamond shaped region between \( x = 0, x = a \) and \( v_0t = a/2, v_0t = -a/2 \). In this region, the solution is
\[ y(t, x) = -\frac{(x - v_0t)^2}{4a} + \frac{(x - v_0t)^3}{6a^2} + \frac{(x + v_0t)^2}{4a} - \frac{(x + v_0t)^3}{6a^2} \] (34)
Notice that the constant \( c \) has cancelled out in this equation. We could have chosen it to be zero from the start.

(d) The velocity of the midpoint of the string is given by
\[ \frac{\partial y(t, a/2)}{\partial t} = \frac{v_0}{4} - \frac{v_0^3 t^2}{a^2} \] (35)
We see that the velocity goes to zero when
\[ t = t_0 = \frac{a}{2v_0}. \] (36)
At this time, the displacement of the string midpoint is
\[ y(t_0, a/2) = \frac{a}{4} - \frac{a}{6} = \frac{a}{12}. \] (37)
This must be the maximum vertical displacement because the initial conditions are symmetric about the midpoint of the string and the midpoint had the largest velocity.

(e) Now we are asked to perform a Fourier decomposition of the string displacement. Evidently the allowed frequencies are
\[ \omega_n = \frac{v_0n\pi}{a}. \] (38)
Since \( h_+(u) \) is a periodic function with period \( 2a \) we can express it as a Fourier series
\[ h_+(u) = \sum_n A_n \exp(i\frac{n\pi}{a} u) \] (39)
so that

\[ y(t, x) = \sum_n A_n \left[ \exp \left( \frac{in\pi}{a} (x - v_0 t) \right) - \exp \left( \frac{in\pi}{a} (x + v_0 t) \right) \right]. \] (40)

We just have to find the amplitudes \( A_n \). They are given by

\[ A_n = \frac{1}{2a} \int_{-a}^{a} du e^{-i \frac{n\pi}{a} u} h_+(u). \] (41)

Performing the integral yields

\[ A_n = -\frac{a}{n^4 \pi^4} (-1 + (-1)^n). \] (42)

We see that only odd harmonics contribute (as you might expect).

3 Problem 3. (Zwiebach problem 4.4)

(a) The boundary conditions at \( x = a \) are that both \( y(x) \) and \( \frac{dy}{dx} \) are continuous at \( x = a \).

(b) First, we’ll consider Neumann boundary conditions. We are interested in the normal modes, so we must solve the equations

\[ \frac{d^2y(x)}{dx^2} + \frac{\mu_1}{T_0} \omega^2 y(x) = 0 \] (43)

for \( x \in (0, a) \) and

\[ \frac{d^2y(x)}{dx^2} + \frac{\mu_2}{T_0} \omega^2 y(x) = 0 \] (44)

for \( x \in (a, 2a) \), subject to Neumann boundary conditions at \( x = 0 \) and \( x = 2a \), and to the conditions of continuity of \( y \) and its first derivative at \( x = a \).

The general solution in \( (0, a) \) is

\[ y(x) = A \cos \left( \omega \sqrt{\frac{\mu_1}{T_0}} x + \phi \right). \] (45)

The Neumann condition is

\[ y'(0) = 0 \Rightarrow \phi = 0. \] (46)
Now we turn to the region \((a, 2a)\). The general solution is

\[
y(x) = B \cos \left(\omega \sqrt{\frac{\mu_2}{T_0}} x + \psi \right).
\]  
(47)

This time the boundary condition is

\[
y'(2a) = 0 \Rightarrow \psi = -2\omega \sqrt{\frac{\mu_2}{T_0}} a.
\]  
(48)

Finally, we must impose the conditions of continuity of \(y\) and its derivative at \(x = a\). In fact, to determine the frequencies of oscillation it is enough to impose continuity of the logarithmic derivative of \(y\) at \(x = a\). This condition is

\[
\sqrt{\mu_1} \tan \omega \sqrt{\frac{\mu_1}{T_0}} a = -\sqrt{\mu_2} \tan \omega \sqrt{\frac{\mu_2}{T_0}} a.
\]  
(49)

This is a transcendental equation for the allowed frequencies \(\omega\).

Next, let’s consider Dirichlet boundary conditions. In this case it’s easy to see that the solutions in the two regions are

\[
y(x) = A \sin \omega \sqrt{\frac{\mu_1}{T_0}} x \text{ in } x \in (0, a)
\]  
(50)

and

\[
y(x) = B \sin \omega \sqrt{\frac{\mu_2}{T_0}} (x - 2a) \text{ in } x \in (a, 2a).
\]  
(51)

Continuity of the logarithmic derivative in this case implies

\[
\sqrt{\mu_1} \cot \omega \sqrt{\frac{\mu_1}{T_0}} a = -\sqrt{\mu_2} \cot \omega \sqrt{\frac{\mu_2}{T_0}} a.
\]  
(52)

Again, this is a transcendental equation to be solved for the allowed frequencies.

(c) Now we take \(\mu_1 = \mu_0\) and \(\mu_2 = 2\mu_0\) and calculate the lowest non-trivial frequency. First, for the Neumann case, the equation to be solved is

\[
\tan \omega \sqrt{\frac{\mu_0}{T_0}} a = -\sqrt{2} \tan \omega \sqrt{\frac{2\mu_0}{T_0}} a
\]  
(53)
This equation must be solved numerically. The lowest non-trivial solution is

\[ \omega = \frac{1.34}{a} \sqrt{\frac{T_0}{\mu_0}}. \]  

(54)

For the Dirichlet case, we must solve

\[ \cot \omega \sqrt{\frac{\mu_0}{T_0}} a = -\sqrt{2} \cot \omega \sqrt{\frac{2\mu_0}{T_0}} a. \]  

(55)

The lowest non-trivial solution is

\[ \omega = \frac{1.26}{a} \sqrt{\frac{T_0}{\mu_0}}. \]  

(56)

(d) This problem with Dirichlet boundary conditions is analogous to the following quantum mechanics problem. Consider a particle in an infinite square well between \( x = 0 \) and \( x = 2a \). Suppose there is a potential \( V(x) \) in the square well given by \( V(x) = 0 \) for \( x \in (0, a) \) and \( V(x) = V_0 \) for \( x \in (a, 2a) \). Then the boundary conditions on the wavefunction would be the same as the boundary conditions in our waves on a string problem. There is an important difference between our problem and the analogous quantum mechanics problem - the wave equation (for waves on a string) is second order in time, while the Schrödinger equation is first order in time.

4 Problem 4. Zwiebach problem 4.6

(a) The variation is given by

\[ \delta S = \int dt \left( \frac{\partial L}{\partial q(t)} \delta q(t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} \delta q(t) \right). \]  

(57)

The condition \( \delta S = 0 \) for any variation \( \delta q(t) \) then yields the Euler-Lagrange equation

\[ \frac{\partial L}{\partial q(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} = 0. \]  

(58)

(b) In this part we work with a Lagrangian density describing a field \( \phi(x) \). The variation of the action is

\[ \delta S = \int d^D x \left( \frac{\partial L}{\partial \phi(x)} \delta \phi(x) - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi(x))} \delta \phi(x) \right). \]  

(59)
Now the condition that this variation vanishes for arbitrary $\delta \phi(x)$ forces the integrand to vanish, and so we find the Euler-Lagrange equation

$$\frac{\partial L}{\partial \phi(x)} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi(x))} = 0$$ (60)

5 Problem 5.

(a) The Lagrangian density for the string bead system is just

$$L = \frac{1}{2} \mu(x) \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2$$ (61)

where the mass density is

$$\mu(x) = \mu_0 + m \delta(x - a).$$ (62)

(b) We can deduce the equation of motion of the string from the Euler-Lagrange equation. Of course, the result is just the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu(x)}{T_0} \frac{\partial^2 y}{\partial t^2}$$ (63)

In the regions $(0, a)$ and $(a, 2a)$ this equation simplifies since the mass density is constant in these region:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu_0}{T_0} \frac{\partial^2 y}{\partial t^2}.$$ (64)

The effect of the bead is to modify the boundary condition at $x = a$. Of course, the string is unbroken, so $y(x)$ is continuous at $x = a$. The first derivative of $y$ is no longer continuous, however. In fact, if we integrate the wave equation over a small region $(a - \epsilon, a + \epsilon)$ around $a$ we find

$$\frac{\partial y(a + \epsilon)}{\partial x} - \frac{\partial y(a - \epsilon)}{\partial x} = m \frac{\partial^2 y(a)}{T_0 \partial t^2},$$ (65)

so there is a discontinuity in the slope of the string at $x = a$.

(c) Now we are asked to find the frequencies of oscillation of the system, with both Neumann and Dirichlet boundary conditions. Since we are working with normal modes, the jump condition (65) becomes

$$\frac{\partial y(a + \epsilon)}{\partial x} - \frac{\partial y(a - \epsilon)}{\partial x} = -\omega^2 \frac{m}{T_0} y(a).$$ (66)
Let’s first work with Neumann boundary conditions. The solutions in the regions away from the bead are

\[ y(x) = A \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} x \right) \quad \text{in } (0, a) \quad (67) \]

\[ y(x) = B \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} (x - 2a) \right) \quad \text{in } (a, 2a). \quad (68) \]

Next, let’s impose continuity at \( x = a \). We find that

\[ A \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right) = B \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right). \quad (69) \]

There are two ways to solve this equation: either \( A = B \) or \( \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right) = 0 \). In the latter case, we can already solve for the frequencies:

\[ \omega_n = (2n + 1) \frac{\pi}{2a} \sqrt{\frac{T_0}{\mu_0}}. \quad (70) \]

Now let’s consider the jump condition. Performing the appropriate differentiations yields

\[ (A + B) \sqrt{\frac{\mu_0}{T_0}} \sin \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right) = m \omega \frac{\sqrt{\mu_0}}{\sqrt{\mu_0}} A \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right). \quad (71) \]

We should consider this equation in light of the two kinds of solution to the continuity equation. First, suppose \( \omega_n = (2n + 1) \frac{\pi}{2a} \sqrt{\frac{T_0}{\mu_0}} \). Then the rhs of the jump equation vanishes so that the derivative is actually continuous. To solve the equation we must choose \( A = -B \). The string displacement is given by

\[ y(x) = A \cos \left( (2n + 1) \frac{\pi x}{2a} \right) \quad (72) \]

for all \( x \in (0, 2a) \). Notice that \( y(a) = 0 \); these solutions correspond to a stationary bead!

Next, consider the other class of solutions to the continuity equation with \( A = B \). The jump condition becomes

\[ 2 \sin \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right) = -\frac{\omega m}{\sqrt{\mu_0 T_0}} \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right). \quad (73) \]
This transcendental equation describes the frequencies of the string when the bead moves.

Next, we consider Dirichlet boundary conditions. The displacement of the string is given by

\[ y(x) = A \sin \left( \omega \sqrt{\frac{\mu_0}{T_0}} x \right) \quad \text{in} \quad (0, a) \]  
(74)

\[ y(x) = B \sin \left( \omega \sqrt{\frac{\mu_0}{T_0}} (x - 2a) \right) \quad \text{in} \quad (a, 2a). \]  
(75)

Again, continuity at \( x = a \) can be imposed in two ways: either \( A = -B \) or \( \omega_n = \frac{n\pi}{a} \sqrt{\frac{T_0}{\mu_0}} \).

The jump condition becomes

\[ (A - B) \sqrt{\frac{\mu_0}{T_0}} \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right) = \frac{\omega m}{T_0} A \sin \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right). \]  
(76)

If we pick \( \omega_n = \frac{n\pi}{a} \sqrt{\frac{T_0}{\mu_0}} \), this reduces to \( A = B \); these solutions again correspond to a stationary bead. On the other hand, if we take \( A = -B \) we find a new condition for the frequencies

\[ 2 \cos \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right) = \frac{\omega m}{\sqrt{\mu_0 T_0}} \sin \left( \omega \sqrt{\frac{\mu_0}{T_0}} a \right). \]  
(77)

(d) We can find the required frequency using the solutions we found in part (c), or just by drawing pictures. For Neumann boundary conditions the frequency is

\[ \omega = \frac{\pi}{2a} \sqrt{\frac{T_0}{\mu_0}} \]  
(78)

while for Dirichlet boundary conditions the frequency is

\[ \omega = \frac{\pi}{a} \sqrt{\frac{T_0}{\mu_0}}. \]  
(79)

The Dirichlet frequency is twice the Neumann frequency.

(e) Since the Schrödinger equation is first order in time while the wave equation is second order, an imaginary frequency solution of the wave equation corresponds to a negative energy solution of the Schrödinger equation. In this case, a negative energy corresponds to a bound state.
With Dirichlet boundary conditions, the quantum mechanics problem really is just a particle in a box of infinite height with a delta function potential in the middle. The bound state wavefunction is a $\sinh$ on either side of the delta function with an appropriate jump in its derivative across the delta function.