1 Problem 1

a) We are given

\[ I(\epsilon) = \sum_{n=1}^{\infty} n e^{-\epsilon n} = -\partial_{\epsilon} \sum_{n=0}^{\infty} e^{-\epsilon n} \]  

Now we can sum the geometric series to find

\[ I(\epsilon) = -\partial_{\epsilon} \frac{1}{1 - e^{-\epsilon}} = \frac{e^{-\epsilon}}{(1 - e^{-\epsilon})^2}. \]

When \( \epsilon \to 0 \), \( I \to \infty \), but for any finite \( \epsilon \), \( I \) is finite. Let us then suppose \( \epsilon \) is very small and expand \( I \) around \( \epsilon = 0 \). We just need to use \( e^{-\epsilon} = 1 - \epsilon + \frac{\epsilon^2}{2} \) (note that we are expanding to second order in \( \epsilon \)).

\[ I(\epsilon) = \frac{1 - \epsilon + \frac{\epsilon^2}{2}}{\epsilon^2(1 - \frac{1}{2} \epsilon + \frac{5}{12} \epsilon^2)^2} \]

\[ = \frac{1}{\epsilon^2} \left(1 - \epsilon + \frac{\epsilon^2}{2}\right) \left(1 + \epsilon + \frac{5}{12} \epsilon^2\right) \]

\[ = \frac{1}{\epsilon^2} \left(1 - \frac{1}{12} \epsilon^2\right) \]

\[ = \frac{1}{\epsilon^2} - \frac{1}{12}. \]

Clearly the first term is divergent as \( \epsilon \to 0 \) so we subtract it. The result is that

\[ I(0) = -\frac{1}{12} \]

b) This problem comes from [1]. We start with

\[ I(x) = \sum_{n=1}^{\infty} (n + x). \]
Notice that
\[ I(x + 1) - I(x) = (2 + x + 3 + x + \cdots) - (1 + x + 2 + x + 3 + x + \cdots) \]
\[ = -(x + 1). \]

Now, we can solve the difference equation by spotting the pattern:
\[ I(1) = I(0) - 1 \]
\[ I(2) = I(0) - 1 - 2 \]
\[ I(3) = I(0) - 1 - 2 - 3, \]
so
\[ I(n) = I(0) - \sum_{m=0}^{n} m = I(0) - \frac{1}{2}n(n + 1). \]

Hence, \( I(x + 1) - I(x) = -(x + 1) \) again.

Now,
\[ I(-\frac{1}{2}) = -\sum_{n=1}^{\infty} (n - \frac{1}{2}) = -\frac{1}{2} \sum_{n=1}^{\infty} (2n - 1). \]

We can also figure out an expression for \( I(0) \) as follows:
\[ I(0) = 1 + 2 + 3 + 4 \cdots \]
\[ 2I(0) = 2 + 4 + 6 + 8 \cdots \]
Subtracting, we find
\[ -I(0) = 1 + 3 + 5 + \cdots = \sum_{n=1}^{\infty} (2n - 1). \]

Note that
\[ I(0) = -2I(\frac{1}{2}) \]
so, using the solution to the recursion relation,
\[ I(-\frac{1}{2}) = \frac{1}{8} + I(0) = -\frac{1}{2}I(0) \]
Solving, we find
\[ I(0) = -\frac{1}{12}. \]

2 Problem 2

a) We need to compute three commutators: \([\alpha_n, \bar{\alpha}_m] \), \([\alpha_n, \alpha_m] \) and \([\bar{\alpha}_n, \bar{\alpha}_m] \). We need the commutator of the \( \alpha_n^I \) operators:
\[ [\alpha_m^I, \alpha_n^I] = m\eta^{ij} \delta_{m+n,0}. \]
Using these, we find
\[
\begin{align*}
[\alpha_n, \bar{\alpha}_m] &= n\delta_{m+n,0} \quad (23) \\
[\alpha_n, \alpha_m] &= 0 \quad (24) \\
[\bar{\alpha}_n, \alpha_m] &= 0. \quad (25)
\end{align*}
\]
We are also asked to express \( S \) and \( m^2 \) in terms of \( \alpha_n \) and \( \bar{\alpha}_n \). The trick is to notice that
\[
\alpha_{-n}\bar{\alpha}_n = \frac{1}{2} \left( \alpha_{-n}^2\alpha_n^2 + \alpha_{-n}^3\alpha_n^3 + i\alpha_{-n}^3\alpha_n^2 - i\alpha_{-n}^2\alpha_n^3 \right), \quad (26)
\]
so that we can construct \( m^2 \) and \( S \) by real and imaginary parts of these. The result is
\[
\begin{align*}
S &= \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_{-n}\bar{\alpha}_n - \bar{\alpha}_{-n}\alpha_n \right) \quad (27) \\
m^2 &= \frac{1}{\alpha'} \left[ -1 + \sum_{n=1}^{\infty} \left( \alpha_{-n}\bar{\alpha}_n + \bar{\alpha}_{-n}\alpha_n \right) \right] \quad (28)
\end{align*}
\]

b) Consider an arbitrary string state \( |\psi\rangle \) which is an eigenstate of \( S \). Then
\[
S|\psi\rangle = s|\psi\rangle \Rightarrow \langle \psi|S|\psi\rangle = s, \quad (29)
\]
where \( s \) is the angular momentum of the state. Hence,
\[
\begin{align*}
s &= \langle \psi|\sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_{-n}\bar{\alpha}_n - \bar{\alpha}_{-n}\alpha_n \right) |\psi\rangle \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left[ \langle \psi|\bar{\alpha}_n^\dagger\alpha_n|\psi\rangle - \langle \psi|\alpha_n^\dagger\alpha_n|\psi\rangle \right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left[ x_n^2 - y_n^2 \right]. \quad (30)
\end{align*}
\]
Notice that \( x_n^2 \) and \( y_n^2 \) are positive real numbers here, since
\[
y_n^2 = \langle \psi|\alpha_n^\dagger\alpha_n|\psi\rangle = ||\alpha_n|\psi\rangle||^2, \quad (33)
\]
for example.

Now, the mass of this state is given by
\[
1 + \alpha' m^2 = \sum_{n=1}^{\infty} \left[ \langle \psi|\bar{\alpha}_n^\dagger\alpha_n|\psi\rangle + \langle \psi|\alpha_n^\dagger\alpha_n|\psi\rangle \right] \\
= \sum_{n=1}^{\infty} \left[ x_n^2 + y_n^2 \right]. \quad (34)
\]
Therefore,
\[
s = \sum_{n=1}^{\infty} \frac{1}{n} \left[ x_n^2 - y_n^2 \right] \leq \sum_{n=1}^{\infty} \left[ x_n^2 + y_n^2 \right] = 1 + \alpha' m^2, \quad (36)
\]
as desired.

Now, we can saturate this inequality by choosing $y_n = 0$ for all $n$ and $x_n = 0$ for $n \geq 2$. But then the state $|\psi\rangle$ must be of the form

$$|\psi\rangle = \frac{1}{\sqrt{N!}}(\alpha^+_1)^N |p^+, \vec{p}_T\rangle = \frac{1}{\sqrt{N!}} \alpha^N_1 |p^+, \vec{p}_T\rangle,$$

where the constant factor is just for normalisation.

c) We are asked to consider the expectation value of $(\Delta X^2)^2 + (\Delta X^3)^2$, which gives a measure of the (squared) length of the string. Notice that

$$\Delta X^2(\tau) = \sum_{n=1}^{\infty} i\sqrt{2\alpha'} \frac{1}{n} (\alpha^n_1 e^{-in\tau} - \alpha_{-n} e^{in\tau}) [(-1)^n - 1].$$

It’s convenient to work with the $\alpha_n$, $\bar{\alpha}_n$ operators, so consider the operators

$$\Delta X = \Delta X^2 + i\Delta X^3 = 2i\sqrt{\alpha'} \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^n_1 e^{-in\tau} - \alpha_{-n} e^{in\tau}) [(-1)^n - 1]$$

$$\Delta \bar{X} = \Delta X^2 - i\Delta X^3 = 2i\sqrt{\alpha'} \sum_{n=1}^{\infty} \frac{1}{n} (\bar{\alpha}_n e^{-in\tau} - \bar{\alpha}_{-n} e^{in\tau}) [(-1)^n - 1].$$

These operators satisfy

$$\Delta X \Delta \bar{X} = (\Delta X^2)^2 + (\Delta X^3)^2,$$

since $X^2$ commutes with $X^3$. We will choose our state $|\psi\rangle = \frac{1}{\sqrt{N!}} \alpha^N_{-1} |p^+, \vec{p}_T\rangle$. Then, the length $\ell$ is

$$\ell^2 = \langle \psi | \Delta X \Delta \bar{X} | \psi \rangle$$

$$= (2i\sqrt{\alpha'})^2 \langle \psi | \left( \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^n_1 e^{-in\tau} - \alpha_{-n} e^{in\tau}) [(-1)^n - 1] \right) \left( \sum_{m=1}^{\infty} \frac{1}{m} (\bar{\alpha}_m e^{-im\tau} - \bar{\alpha}_{-m} e^{im\tau}) [(-1)^m - 1] \right) | \psi \rangle$$

$$= 4\alpha' \langle \psi | \sum_{n=1}^{\infty} \frac{1}{n^2} \alpha^n \bar{\alpha}_{-n} [(-1)^n - 1]^2 | \psi \rangle$$

$$= 16\alpha' \langle \psi | \alpha_1 \bar{\alpha}_{-1} | \psi \rangle$$

$$= 16\alpha' (1 + \alpha'm^2).$$

Therefore, $\ell^2 = 16\alpha'^2 m^2 + 16\alpha'$. This is just the classical equation $\ell^2 = 16\alpha'^2 m^2$ with a quantum correction $16\alpha'$.

d) Now we consider one of these string states with large $s$. Suppose it decays into two string states. Conservation of angular momentum tells us that $s = s_1 + s_2$. Since $s_1 \leq N_1$ and $s_2 \leq N_2$ we have

$$s = N = s_1 + s_2 \leq N_1 + N_2.$$
We must also conserve energy. In particular, the mass of the initial string must be big enough that the two decay products have smaller summed mass than the initial string, so \( m \geq m_1 + m_2 \). Using \( \alpha' m^2 + 1 = N \), this becomes

\[
N \geq N_1 + N_2 - 1 + 2\sqrt{N_1 - 1}\sqrt{N_2 - 1}.
\]  

(48)

Putting the two inequalities together we find

\[
N_1 + N_2 - 1 + 2\sqrt{N_1 - 1}\sqrt{N_2 - 1} \leq N \leq N_1 + N_2.
\]  

(49)

Therefore,

\[
2\sqrt{N_1 - 1}\sqrt{N_2 - 1} \leq 1.
\]  

(50)

We must solve this inequality over the positive integers. Since \( N \) is large and \( N_1 + N_2 \geq N \) at least one of \( N_1 \) or \( N_2 \) must be a large number. Then it is evident that to solve our final inequality we must have either \( N_1 = 1 \) or \( N_2 = 1 \). Let’s just choose to take \( N_2 = 1 \). We just have to figure out what \( N_1 \) can be. Going back to our inequalities with \( N_2 = 1 \), we find

\[
N_1 \leq N \leq N_1 + 1.
\]  

(51)

There are two possible solutions: either \( N_1 = N - 1 \) or \( N_1 = N \). But we can rule out \( N_1 = N \) since this is forbidden kinematically; in general, a massive state cannot emit another particle without decaying (unless the other particle is a tachyon; in our case the other particle is massless.) Therefore, we must choose \( N_1 = N - 1 \).

We have shown that a leading Regge trajectory string can only decay by emitting a string with \( N = 1 \). This string is a massless gauge boson (photon) and must be circularly polarised in the \( x^2, x^3 \) plane by conservation of angular momentum. It just remains to compute the energy of the photon. We will work in the rest frame of the initial string.

First, notice that the mass \( m_1 \) of the massive decay product is

\[
1 + \alpha' m_1^2 = N_1 = N - 1 = \alpha' m^2 \Rightarrow m_1^2 = m^2 - \frac{1}{\alpha'}.
\]  

(52)

Let’s call the energy-momentum of the photon \((k, \mathbf{k})\) and of the new string \((E, \mathbf{p})\). Conservation of energy implies

\[
(E, \mathbf{p}) = -(m, \mathbf{0}) + (k, \mathbf{k}).
\]  

(53)

Squaring this equation gives

\[
m_1^2 = m^2 - 2mk \Rightarrow k = \frac{1}{2\alpha'm}.
\]  

(54)

This problem is discussed in [2].

### 3 Problem 3

a) Figure 1 is a picture of a string before the twist and after the twist (dashed.) The strings are at the same point in space but their orientation (the arrow in the figure) is reversed.
b) The definition of the twist is

$$\Omega X^I(\tau, \sigma)\Omega^{-1} = X^I(\tau, \pi - \sigma).$$  \hfill (55)

To figure out the action of $\Omega$ on the oscillators, we use the mode expansion

$$X^I(\tau, \sigma) = x^I_0 + \sqrt{2\alpha'}\alpha^I_0 \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha^I_n \cos n\sigma e^{-in\tau}. \hfill (56)$$

Acting with $\Omega$ we find

$$X^I(\tau, \pi - \sigma) = x^I_0 + \sqrt{2\alpha'}\alpha^I_0 \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha^I_n (-1)^n \cos n\sigma e^{-in\tau} \hfill (57)$$

so

$$\Omega x^I_0\Omega^{-1} = x^I_0 \hfill (58)$$
$$\Omega \alpha^I_n\Omega^{-1} = (-1)^n \alpha^I_n. \hfill (59)$$

c) We need the mode expansion for $X^-$ in terms of the Virasoro operators:

$$X^-(\tau, \sigma) = x^-_0 + \sqrt{2\alpha'}\alpha^-_0 \tau + i\frac{1}{p^+} \sum_{n \neq 0} \frac{1}{n} L^-_n \cos n\sigma e^{-in\tau}, \hfill (60)$$
where

\[ L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^j \alpha_p^j. \]  

(61)

Notice that

\[ \Omega L_n^\perp \Omega^{-1} = (-1)^n L_n^\perp. \]  

(62)

Hence,

\[ \Omega X^{-}(\tau,\sigma)\Omega^{-1} = X^{-}(\tau,\pi - \sigma). \]  

(63)

It’s clear that the Hamiltonian is invariant, either by reparametrization invariance or by remembering that \( H = L_0^\perp - 1. \)

d) Here is a table of the open string states with \( N^\perp \leq 3 \) and their twist eigenvalues:

<table>
<thead>
<tr>
<th>( N^\perp )</th>
<th>State</th>
<th>Twist</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
<tr>
<td>1</td>
<td>(\alpha_{-1}</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
<tr>
<td>2</td>
<td>(\alpha_{-1}\alpha_{-1}</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
<tr>
<td></td>
<td>(\alpha_{-2}</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
<tr>
<td>3</td>
<td>(\alpha_{-1}\alpha_{-1}</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
<tr>
<td></td>
<td>(\alpha_{-2}\alpha_{-1}</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
<tr>
<td></td>
<td>(\alpha_{-3}</td>
<td>p^+, \vec{p}_T\rangle)</td>
</tr>
</tbody>
</table>

We can prove that the twist eigenvalue of a state \(|\psi\rangle\) is \((-1)^{N^\perp}\) by considering a general state

\[ |\psi\rangle = \prod_k \alpha_{-m_k}|p^+, \vec{p}_T\rangle. \]  

(64)

Notice that this state has \( N^\perp = \sum_k m_k. \) Meanwhile, it has twist eigenvalue

\[ \Omega|\psi\rangle = \prod_k (\Omega \alpha_{-m_k}\Omega^{-1})|p^+, \vec{p}_T\rangle \]  

(65)

\[ = \prod_k ((-1)^m\alpha_{-m_k})|p^+, \vec{p}_T\rangle \]  

(66)

\[ = (-1)^{\sum_i m_i} \prod_k \alpha_{-m_k}|p^+, \vec{p}_T\rangle \]  

(67)

\[ = (-1)^{N^\perp}|\psi\rangle \]  

(68)

e) The states which are invariant under the twist are evidently those with twist eigenvalue +1. From part (d) we know these states have even (or zero) \( N^\perp. \) Thus, we must discard the states with odd \( N^\perp. \) We can do this by projecting onto the subspace of the full Hilbert space with \( \Omega = +1. \) The projector is just

\[ P = \frac{1}{2} (1 + \Omega). \]  

(69)
References
