1 Problem 1

a) The expansion of the potential is

\[ V(\phi) = \frac{\lambda}{2}(\phi^4 - 2v^2\phi^2 + v^4) \]  

(1)

A real scalar has a mass term like \( \frac{1}{2}m^2\phi^2 \) in the potential, so we recognise that the mass in this case is \(-2\lambda v^2\).

b) Evidently, if we consider static fields, the Lagrangian reduces to

\[ \mathcal{L} = -\int dx \left( \frac{1}{2}(\partial_x \phi)^2 + V(\phi) \right) . \]  

(2)

This is just the negative of the action for a one dimensional system with potential \(-V\). The overall sign is unimportant, so we can drop it. Hence, the Hamiltonian of the system is

\[ H = \frac{1}{2}(\partial_x \phi)^2 - V(\phi) . \]  

(3)

since the Hamiltonian is conserved, this is a constant, \(E\). Therefore, we have a differential equation for the solution \(\phi\):

\[ \partial_x \phi = \pm \sqrt{2(E + V(\phi))} . \]  

(4)

Now, we have boundary conditions that \(\phi(x \to -\infty) = -v\) and \(\phi(x \to \infty) = v\). Therefore, as \(x \to \infty\), \(V \to 0\) and \(\partial_x \phi \to \infty\). Inserting these conditions in our differential equation, we find \(E = 0\). Therefore we have,

\[ \partial_x \phi = \pm \sqrt{2V(\phi)} = \pm \sqrt{\lambda} (\phi^2 - v^2) . \]  

(5)

The solution to this equation that matches the boundary conditions (we have to choose the \(-\) sign) is

\[ \phi(x) = v \tanh(\sqrt{\lambda}x) . \]  

(6)
2 Problem 2

a) We need the oscillator expansion of the string coordinates:

\[ X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'} \alpha^\mu_0 \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{i n \tau}}{n} (\alpha_n^{\mu} e^{i n \sigma} + \bar{\alpha}_n^{\mu} e^{-i n \sigma}). \]  

(7)

Now, we know that \( X(\tau, \sigma') = X(\tau, \sigma + \sigma_0) \), so we can find the primed oscillators in terms of the oscillators defined above:

\[ X^\mu(\tau, \sigma') = x_0^\mu + \sqrt{2\alpha'} \alpha^\mu_0 \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{i n \tau}}{n} (\alpha_n^{\mu} e^{i n(\sigma + \sigma_0)} + \bar{\alpha}_n^{\mu} e^{-i n(\sigma + \sigma_0)}). \]

(8)

Comparing these, we find

\[ \alpha_n^{\mu} = \alpha_n^{\mu} e^{i n \sigma_0} \]

(10)

\[ \bar{\alpha}_n^{\mu} = \bar{\alpha}_n^{\mu} e^{-i n \sigma_0} \]

(11)

b) Using Zwiebach 13.26, we have

\[ e^{-i P_{\sigma_0}} \left( \dot{X}^I(\tau, \sigma) + X'^{I}(\tau, \sigma) \right) e^{i P_{\sigma_0}} = \dot{X}^I(\tau, \sigma + \sigma_0) + X'^{I}(\tau, \sigma + \sigma_0) \]

(12)

\[ = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \bar{\alpha}_n^{\mu} e^{-i n(\tau + \sigma + \sigma_0)} \]

(13)

\[ = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} e^{-i P_{\sigma_0}} \bar{\alpha}_n^{\mu} e^{i P_{\sigma_0}} e^{-i n(\tau + \sigma)}, \]

(14)

so

\[ e^{-i P_{\sigma_0}} \bar{\alpha}_n e^{i P_{\sigma_0}} = \bar{\alpha}_n e^{-i n \sigma_0} = \bar{\alpha}_n' \]

(15)

Similarly, we find that

\[ e^{-i P_{\sigma_0}} \alpha_n e^{i P_{\sigma_0}} = \alpha_n e^{i n \sigma_0} = \alpha_n' \]

(16)

c) We act on \( \ket{\psi} \) with the operator \( e^{-i P_{\sigma_0}} \), and require that the operator acts as
the identity since the state should be invariant. Therefore,

\[ |\psi\rangle = e^{-iP\sigma_0}|\psi\rangle \]

(17)

\[ = e^{-iP\sigma_0} \left( \prod_{p=1}^{N} \bar{\alpha}_{-n_p} \right) \left( \prod_{q=1}^{M} \alpha_{-m_q} \right) |p^+, p^I\rangle \]

(18)

\[ = e^{-iP\sigma_0} \left( \prod_{p=1}^{N} \bar{\alpha}_{-n_p} \right) e^{iP\sigma_0} e^{-iP\sigma_0} \left( \prod_{q=1}^{M} \alpha_{-m_q} \right) |p^+, p^I\rangle \]

(19)

\[ = e^{-i\sigma_0} \left( \sum_{p=1}^{N} n_p - \sum_{q=1}^{M} m_q \right) \left( \prod_{p=1}^{N} \bar{\alpha}_{-n_p} \right) \left( \prod_{q=1}^{M} \alpha_{-m_q} \right) |p^+, p^I\rangle \]

(20)

\[ = e^{-i\sigma_0} \left( \sum_{p=1}^{N} n_p - \sum_{q=1}^{M} m_q \right) |\psi\rangle. \]

(21)

Therefore, invariance requires that

\[ \sum_{p=1}^{N} n_p = \sum_{q=1}^{M} m_q. \]

(22)

3 Problem 3 (Zwiebach problem 13.6)

(This solution is taken from the course given last year.)

a) The O23 plane is described by

\[ \Omega_{p} X^{24,25}(\tau, \sigma) \Omega^{-1} = -X^{24,25}(\tau, 2\pi - \sigma) \]

and all others directions satisfying \( \Omega_{p} X^{i}(\tau, \sigma) \Omega^{-1} = X^{i}(\tau, 2\pi - \sigma) \). In Figure (2) we draw a single closed oriented string, \( X^{\mu}(\tau, \sigma) \), in the first quadrant with a continuos line and we draw the string \( \tilde{X}^{a}(\tau, \sigma) = -X^{a}(\tau, 2\pi - \sigma) \) with a dashed line.

b) From the mode expansion (13.24) we have that

\[ \Omega_{p} x_{0}^{a} \Omega^{-1} = -x_{0}^{a} \quad \text{and} \quad \Omega_{p} p^{a} \Omega^{-1} = -p_{0}^{a} \]

\[ \Omega_{p} \alpha_{n}^{a} \Omega^{-1} = -\alpha_{n}^{a} \quad \text{and} \quad \Omega_{p} \bar{\alpha}_{n}^{a} \Omega^{-1} = -\bar{\alpha}_{n}^{a} \]

and similarly

\[ x_{0}^{i} \rightarrow x_{0}^{i} \quad \text{and} \quad p^{j} \rightarrow p^{j} \]

\[ \bar{\alpha}_{n}^{j} \rightarrow \bar{\alpha}_{n}^{j} \quad \text{and} \quad \bar{\alpha}_{n}^{j} \rightarrow \bar{\alpha}_{n}^{j}. \]
Figure 1: The string $X^\mu(\tau,\sigma)$ is drawn with a continuous line while an image, reflected in $X^{24}$, $X^{25}$ axis and reversed in orientation is drawn with a dashed line. The arrow points in the direction of increasing $\sigma$.

To relate the $X^-$ coordinate to the transverse directions we again use the relation

$$\dot{X}^- \pm X^- = \frac{1}{\alpha'} \frac{1}{2p^+} \left( \dot{X}^I \pm X^I' \right)^2$$

and we note that because of the square the minus signs cancel

$$\Omega_p(\dot{X}^- \pm X^-)\Omega^{-1} = \frac{1}{\alpha'} \frac{1}{2p^+} \Omega_p \left( \dot{X}^I \pm X^I' \right) \Omega^{-1}_p \Omega_p \left( \dot{X}^I \pm X^I' \right) \Omega^{-1}_p$$

$$= \frac{1}{\alpha'} \frac{1}{2p^+} \left( \dot{X}^I \pm X^I' \right)^2 (\tau, 2\pi - \sigma).$$

For $X^+$ we can use light-cone gauge, $X^+ = \alpha' p^+ \tau$, to see that

$$\Omega_p X^+\Omega^{-1}_p = X^+(\tau, \sigma)$$

$$= X^+(\tau, 2\pi - \sigma).$$

The orientifold transformations will leave the closed string Hamiltonian invariant and hence are symmetries of the theory.

c) The state $|p^+, p^i, p^\alpha\rangle$ satisfies $\tilde{p}^\mu |p^+, p^i, p^\alpha\rangle = p^\mu |p^+, p^i, p^\alpha\rangle$. So we now consider $\tilde{p}^\alpha \Omega_p |p^+, p^i, p^\alpha\rangle$ with can be written as

$$\tilde{p}^\alpha \Omega_p |p^+, p^i, p^\alpha\rangle = \Omega_p (\Omega^{-1}_p \tilde{p}^\alpha \Omega_p) |p^+, p^i, p^\alpha\rangle$$
using $\Omega_p\Omega_p = 1 \Rightarrow \Omega_p = \Omega_p^{-1}$. Hence
\[
\hat{p}_a^\dagger \Omega_p |p^+, p^i, p^a\rangle = -\Omega_p \hat{p}_a^\dagger |p^+, p^i, p^a\rangle = -p_a(\Omega_p |p^+, p^i, p^a\rangle)
\]

So
\[
\Omega_p |p^+, p^i, p^a\rangle = |p^+, p^i, -p^a\rangle.
\]

We could also start with,
\[
|p^a\rangle = e^{ip^a\hat{x}_a}|0\rangle
\]

so that
\[
\Omega_p |p^a\rangle = \Omega_p e^{ip^a\hat{x}_a}\Omega_p^{-1} |0\rangle = e^{-ip^a\hat{x}_a}|0\rangle, \text{ as } \Omega_p |0\rangle = |0\rangle = |0\rangle.
\]

d) To find the conditions which the wavefunctions must satisfy in order to guarantee $\Omega_p$ invariance consider

\[
\Omega_p \Phi = \int dp^+ d\vec{p}^i d\vec{p}^a [\Phi_+^{ij}(\tau, p^+, p^i, p^a)(\bar{\alpha}_{-1}^i \alpha_{-1}^j \pm \bar{\alpha}_{-1}^j \alpha_{-1}^i |p^+, p^i, -p^a\rangle \\
- \Phi_-^{ij}(\tau, p^+, p^i, p^a)(\bar{\alpha}_{-1}^i \alpha_{-1}^j \pm \bar{\alpha}_{-1}^j \alpha_{-1}^i |p^+, p^i, -p^a\rangle) \\
+ \Phi_+^{ab}(\tau, p^+, p^i, p^a)(\bar{\alpha}_{-1}^a \alpha_{-1}^b \pm \bar{\alpha}_{-1}^b \alpha_{-1}^a |p^+, p^i, -p^a\rangle)]
\]

\[
= \int dp^+ d\vec{p}^i d\vec{p}^a (-1)^a [\pm \Phi_{+ij}^{a}(\tau, p^+, p^i, -p^a)(\alpha_{-1}^i \bar{\alpha}_{-1}^j \pm \alpha_{-1}^j \bar{\alpha}_{-1}^i |p^+, p^i, -p^a\rangle \\
\mp \Phi_{ia}^{a}(\tau, p^+, p^i, -p^a)(\alpha_{-1}^i \bar{\alpha}_{-1}^a \pm \alpha_{-1}^a \bar{\alpha}_{-1}^i |p^+, p^i, -p^a\rangle) \\
\pm \Phi_{ab}^{a}(\tau, p^+, p^i, -p^a)(\alpha_{-1}^b \bar{\alpha}_{-1}^a \pm \alpha_{-1}^a \bar{\alpha}_{-1}^b |p^+, p^i, -p^a\rangle)]
\]

So for invariance we must have
\[
\Phi_{ij}^{a}(\tau, p^+, p^i, p^a) = \pm (-1)^a \Phi_{ij}^{a}(\tau, p^+, p^i, -p^a) \\
\Phi_{ia}^{a}(\tau, p^+, p^i, p^a) = \mp (-1)^a \Phi_{ia}^{a}(\tau, p^+, p^i, -p^a) \\
\Phi_{ab}^{a}(\tau, p^+, p^i, p^a) = \pm (-1)^a \Phi_{ab}^{a}(\tau, p^+, p^i, -p^a)
\]
4 Problem 4 (Zwiebach problem 15.4)

We will solve this problem by exploiting the analogy to magnetostatics. The string is like an infinitely long current carrying conductor; therefore we can find the field $\vec{B}_H$ using Ampere’s Law. We can calculate the field strength $H$ from this information. The potential $B$ for $H$ will turn out to simply be the vector potential associated with the “magnetic” field $\vec{B}_H$.

To get started, suppose the string is lying along the $x$-axis. The current density is then $\vec{j}^0 = \frac{1}{2}\delta(y)\delta(z)$. We will find $\vec{B}_H$ by exploiting the symmetry of the problem. By Ampere’s Law, integrating around a circle a distance $\rho$ in the $yz$-plane from the wire,

$$\frac{1}{\kappa^2} 2\pi \rho B_H = \frac{1}{2} \Rightarrow B_H = \frac{\kappa^2}{4\pi \rho}.$$  \hfill (23)

The direction of the field is given by the right-hand rule. That is,

$$\vec{B}_H = -\frac{\kappa^2}{4\pi \rho^2} z\hat{y} + \frac{\kappa^2}{4\pi \rho^2} y\hat{z}.$$  \hfill (24)

Since $h^{0kl} = \epsilon^{klm} B_{Hm}$, we can find $H$. The only non-zero components are

$$H^{013} = \frac{\kappa^2}{4\pi \rho^2} z$$  \hfill (25)

$$H^{012} = \frac{\kappa^2}{4\pi \rho^2} y,$$  \hfill (26)

(of course, components of $H$ related to these by antisymmetry are non-zero, too.)

Now we will find $B$. In this time independent situation, the relationship between $H$ and $B$ simplifies. We find that $B^{ij} = 0$ since $H^{ijk} = 0$, so

$$H^{0ij} = \partial_i B^{0j} - \partial_j B^{0i}.$$  \hfill (27)

We can solve these equations by taking $B^{02} = B^{03} = 0$ and

$$B^{01} = -\frac{\kappa^2}{8\pi} \log \left( \frac{y^2 + z^2}{L^2} \right)$$  \hfill (28)

where $L$ is an arbitrary length scale.

The physics of this is simple. Take $\vec{B}^0 = \vec{A}$ to be the vector potential of some magnetostatic configuration. Then a simple calculation shows that $\nabla \times \vec{A}$ is the magnetic field $\vec{B}_H$ we encountered above.