The Classical Static Gauge String

Required reading: Zwiebach §6.4, §7.1-3

Suggested reading:
Zwiebach §6.8
Schwarz & Schwarz §10.4 (3rd subsection)
Polchinski §1.2

Open and closed strings:
When it comes to relativistic strings, we have to consider two types.

- *Open strings*, which we know from the nonrelativistic case. These have two endpoints.
- *Closed strings* have no endpoints and form a complete loop. Since an interacting open string can produce a closed string, we can never have a theory of just open strings. On the other hand, closed strings are fine all by themselves.

I won’t mention oriented vs unoriented strings just yet.

An open string on the left and a closed string on the right.

The Nambu-Goto action:
Once again, what kind of action should we have? For a relativistic string, we should use a relativistic invariant. Let’s just use the “area” for the surface of the string:

\[ S = -T_0 \int d^2\xi \sqrt{-\det g_{\alpha\beta}} = -T_0 \int d^2\xi \sqrt{\det (g_{\mu\nu} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu)} . \]

For strings, this is called the Nambu-Goto action. To get a dimensionless action, note that \( T_0 \) is a tension with units of \( [M]^2 \) since the area has units \( [M]^{-2} \). Henceforth, let’s specialize to Minkowski spacetime for simplicity. The symmetries of this action are

- Poincaré invariance (Lorentz and translations) (or coordinate transformation invariance in more general spacetimes).
- Reparameterization of \( \xi^0, \xi^1 \).
If we want to, we can generalize this action nicely to any dimension object, as long as we can describe it in terms of a surface in spacetime. In string theory, D-branes have a variety of dimensions, and their action is based on this Nambu-Goto form.

**Static gauge:**

Here we will make a choice of parameterization for $\xi^\alpha$. This really is like choosing a gauge in electrodynamics, where we choose a particular form of potential and vector potential to represent the electromagnetic fields, even though other choices may work.

- Begin by considering the string *worldsheet*, the surface that the string sweeps out in its motion through spacetime. In the worldsheet coordinates $\xi^\alpha$, it’s just a Cartesian grid, but it’s generally “wavy” in spacetime.

![Worldsheet view](image1)

![Spacetime view](image2)

The coordinate grid on the worldsheet, seen from the perspective of the worldsheet and the spacetime embedding. Horizontal lines are constant $\xi^0$, and vertical lines are constant $\xi^1$.

- Make sure we’ve chosen a particular Lorentz frame. Then we can define $\xi^0 = t$ for each point on the string, where $t$ is the coordinate time in that frame. In the figure, then, the constant $\xi^0$ lines lie entirely in constant time hyperplanes.

- This choice is the *static gauge*. Recall from our study of the point particle that this choice means $X^0(\xi)$ is no longer a degree of freedom of the string.

- The tangents to the worldsheet now are

$$\partial_0 X^\mu = [1, \partial_1 \vec{X}] \; , \; \partial_1 X^\mu = [0, \partial_0 \vec{X}] \; .$$
We have defined $\xi^1 = \sigma$ just to give it a name.

- The action is now
  \[
  S = -T_0 \int d^2\xi \sqrt{(\partial_0 X^\mu \partial_1 X_\mu)^2 - (\partial_0 X)^2 (\partial_1 X)^2} \\
  = -T_0 \int d^2\xi \sqrt{1 - |\partial_1 X|^2 |\partial_\sigma X|^2 + (\partial_1 X \cdot \partial_\sigma X)^2}.
  \]

- We still get to make a choice of $\sigma$ parameterization. Without justifying our choice just yet, let’s require two constraints to be satisfied everywhere on the worldsheet:
  \[
  \partial_1 X \cdot \partial_\sigma X = 0 \quad (1) \\
  |\partial_1 X|^2 + |\partial_\sigma X|^2 = 1. \quad (2)
  \]

- We can use the constraints to get rid of the square root, if we’re clever. The second term in the radical vanishes due to the first constraint, and the second constraint lets us write each factor of the first term as $(1 - |\partial_1 X|^2 + |\partial_\sigma X|^2)/2$. Then
  \[
  S = -\frac{T_0}{2} \int d^2\xi \left(1 - |\partial_1 X|^2 + |\partial_\sigma X|^2\right).
  \]

- It’s relatively much easier to get the equation of motion now. It’s just the wave equation
  \[
  \partial_\sigma^2 X - \partial_t^2 X = 0.
  \]

- Note that we’ve made the action look just like that for the nonrelativistic string up to an additive constant. Also, the mass density of the string is the same as the tension, meaning that waves along the string move at the speed of light.

**Example using this static gauge:**

Consider a closed string of radius $\ell$ initially at rest. For simplicity, take it to be circular centered on the origin of the $x, y$-plane. Let’s write out the coordinates in static gauge.

- We have already that $\xi^0 = t$.
- Since the string is initially at rest, $\partial_t X \cdot \partial_\sigma X = 0$ automatically.
- Then we need $(\partial_\sigma X)^2 + (\partial_\sigma Y)^2 = 1$. We can achieve that by taking initial configuration
  \[
  X(t = 0, \sigma) = \ell \cos \left(\frac{2\pi \sigma}{\sigma_1}\right), \quad Y(t = 0, \sigma) = \ell \sin \left(\frac{2\pi \sigma}{\sigma_1}\right)
  \]
  with $\sigma_1 = 2\pi \ell$. 

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• Now we separate variables and guess that each position \( X^i(t, \sigma) = X^i_0(t)X^i_\sigma(\sigma) \) and that \( X^i_0(0) = 1 \) (with initial condition \( \partial_t X^i_0(0) = 0 \)).

• Plugging through the wave equation, we find

\[
X(t, \sigma) = \ell \cos \left( \frac{t}{\ell} \right) \cos \left( \frac{\sigma}{T} \right), \quad Y(t, \sigma) = \ell \cos \left( \frac{t}{\ell} \right) \sin \left( \frac{\sigma}{T} \right).
\]

• It’s straightforward algebra to check that the gauge conditions are satisfied at all times.

**Justifying our \( \sigma \) gauge:**

We already saw explicitly how to make our static gauge choice \( \xi^0 = t \). Now we need to go back and work out how to get the constraints we want by choosing \( \sigma \) carefully.

• First, let’s consider the 1D curve the string traces out at a fixed time \( t_0 \) and choose any arbitrary way to label its points monotonically, starting at \( \sigma = 0 \) and ending at \( \sigma = \sigma_1 \). (On a closed string, the points 0 and \( \sigma_1 \) are the same.)

• Now also consider the curve of the string at \( t_1 = t_0 + \delta t \). Then draw perpendicular segments from the first string to the second; this tells you how to label \( \sigma \) at time \( t_1 \). Repeat this process so that all times are covered.

The two strings and perpendiculars joining points of the same \( \sigma \) value.

• Now, at fixed \( \xi^0, \partial_\sigma \vec{X} \) are tangents and \( \partial_t \vec{X} \) are normals, so we get our first constraint (1).

• On the homework, you will prove that

\[
\partial_t \left( \frac{|\partial_\sigma \vec{X}|}{\sqrt{1 - |\partial_t \vec{X}|^2}} \right) = 0,
\]

which follows from Nöther’s theorem for field theories. It is valid for any choice of \( \sigma \) that satisfies (1).
Now we can choose a reparameterization $\sigma'(\sigma)$ such that

$$\partial_\sigma \sigma' = \frac{|\partial_\sigma \vec{X}|}{\sqrt{1 - |\partial_\sigma \vec{X}|^2}}.$$ 

This implies (2) using the new coordinate $\sigma'$.

**Polyakov action:**

Getting away from Zwiebach a little, let’s look at an alternative action for a string:

$$S = -\frac{T_0}{2} \int d^2 \xi \sqrt{-\det \gamma_{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu}.$$ 

This action is known as the *Polyakov action* for Brink, Di Vecchia, Howe, Deser, and Zumino (actually because Polyakov popularized it). Here, $\gamma_{\alpha\beta}$ is not the induced metric but rather an arbitrary metric for the worldsheet. The symmetries of the Polyakov action are

- Poincaré invariance (or coordinate transformation invariance) again.
- Worldsheet coordinate reparameterization invariance again.
- Weyl invariance, by which we mean taking
  $$\gamma_{\alpha\beta} \to \gamma'_{\alpha\beta} = \Omega(\xi)\gamma_{\alpha\beta}.$$ 

The Polyakov action is popular because it doesn’t have a square root, which makes it much easier to quantize than the Nambu-Goto action. Also, by solving the EOM for $\gamma_{\alpha\beta}$, you can prove that the Polyakov action is classically equivalent to the Nambu-Goto action.

- To start, we’ll need the formula
  $$\delta \det \gamma_{\alpha\beta} = -\det \gamma_{\gamma\delta} \gamma_{\alpha\beta} \delta \gamma^\gamma_{\delta},$$
  which can be derived starting from several equivalent definitions of the determinant. My favorite is writing the inverse of the determinant as
  $$\det \gamma^{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\gamma\beta\delta} \gamma^{\alpha\gamma} \gamma^{\beta\delta}, \quad \epsilon_{00} = \epsilon_{11} = 0, \quad \epsilon_{01} = -\epsilon_{10} = 1.$$ 

  This works for a 2D metric and generalizes easily to any dimension.

- So then you can work out the equation of motion to be
  $$\frac{\delta S}{\delta \gamma_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \gamma_{\alpha\beta} \gamma^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu = 0.$$
• Since the derivative bit is the induced metric, we find

\[ g_{\alpha \beta} = \frac{1}{2} \gamma_{\alpha \beta} \gamma^\gamma g_{\gamma \delta} . \]

This shows that \( \gamma_{\alpha \beta} \) is the induced metric up to a function of \( \xi \), which we can remove by Weyl invariance.

• Plugging back in \( \gamma_{\alpha \beta} = g_{\alpha \beta} \), you can find the Nambu-Goto action.

Another interesting point:

• A 2D metric can always be written, after coordinate transformation, as \( \gamma_{\alpha \beta} = \Omega(\xi) \eta_{\alpha \beta} \). That’s just because we have two independent coordinates to transform and want to impose two conditions on the metric (\( \gamma_{01} = 0 \) and \( \gamma_{00} = -\gamma_{11} \)).

• In these coordinates, the Polyakov action is

\[ S = -\frac{T_0}{2} \int d^2 \xi \eta^{\alpha \beta} \eta_{\mu \nu} \partial_\alpha X^\mu \partial_\beta X^\nu . \]

• However, if we are also allowed to take \( \xi^0 = X^0 \), we get the final static gauge Nambu-Goto action.

What I want you to take away from this discussion is that there is a different form of the action which is usually more useful in a quantum context, and you’ll see it if you pursue string theory further. However, we won’t use it again.