Ph230b, Winter 2011
Problem Set 3 Solution

Hee-Joong Chung

Homework 3

Charged pion decay: In order to incorporate charged pions $\pi^{\pm}$ in the Fermi theory of weak interactions, it is natural to add a term $f_\pi p_\mu \pi^{\pm}$ to the charged current J_μ^- (remember that pions couple derivatively). Here, p_μ is the pion momentum and f_π is the pion decay constant. This leads to the following interaction Lagrangian:

$$\mathcal{L}_\pi = \frac{G_F}{\sqrt{2}} f_\pi p_\rho \pi^{\pm} \overline{\mu} \gamma^\rho (1-\gamma_5)\nu_\mu + h.c.$$

Show that the total rate for $\pi^- \to \mu^- \overline{\nu}_\mu$ decay is

$$\Gamma(\pi^- \to \mu^- \overline{\nu}^\mu) = \frac{G_F^2}{8\pi} f_\pi^2 m_{\pi^-} m_\mu^2 \left(1 - \frac{m_\mu^2}{m_{\pi^-}^2}\right)^2$$

where m_{π^-} (resp. m_μ) is the pion (resp. muon) mass.

Kaon decay: Predict the ratio of the $K^- \to e^- \overline{\nu}_e$ and $K^- \to \mu^- \overline{\nu}_\mu$ decay rates. Given that the lifetime of the K^- is $\tau = 1.2 \times 10^{-8}$ sec and the $K^- \to \mu^- \overline{\nu}_\mu$ branching ratio is 64%, estimate the decay constant f_K.

Solution

Chared pion decay

From the Lagrangian we obtain the scattering amplitude;

$$\mathcal{T} = \frac{G_F}{\sqrt{2}} f_\pi p_\rho \overline{u} \gamma^\rho (1-\gamma_5) v,$$

where $\overline{u}, v$ are four-component spinor of μ^- and $\overline{\nu}_\mu$. From this, we calculate $|\mathcal{T}|^2$, and by summing over final spins, we get

$$\sum_{\text{final spins}} \left|\mathcal{T}^2\right| = \frac{G_F}{2} f_\pi^2 \mathrm{Tr}\left[(-\not{p}_\mu + m_\mu)\gamma^\rho(1-\gamma_5)(-\not{p}_{\overline{\nu}_\mu})\gamma^\delta(1-\gamma_5)\right] p_\rho p_\delta \tag{1}$$

After some calculation, eq(1) becomes

$$\sum_{\text{final spins}} \left|\mathcal{T}^2\right| = 4G_F^2 f_\pi^2 \left(2(p_\mu \cdot p_{\pi^-})(p_{\pi^-} \cdot p_{\overline{\nu}_\mu}) - (p_\mu \cdot p_{\overline{\nu}_\mu})(p_{\pi^-} \cdot p_{\pi^-})\right). \tag{2}$$

In the center of mass frame, $p_{\pi^-} = (m_{\pi^-}, 0)$, $p_\mu = (E_\mu, \mathbf{p})$, and $p_{\overline{\nu}_\mu} = (|\mathbf{p}|, -\mathbf{p})$ where $E_\mu^2 = m_\mu^2 + \mathbf{p}^2$. From $m_{\pi^-} = E_\mu + |\mathbf{p}|$, we obtain $|\mathbf{p}| = \frac{m_{\pi^-}^2 - m_\mu^2}{2m_{\pi^-}}$, $E_\mu = \frac{m_{\pi^-}^2 + m_\mu^2}{2m_{\pi^-}}$. Then eq(2) becomes

$$2G_F^2 f_\pi^2 m_\mu^2 m_{\pi^-}^2 \left(1 - \frac{m_\mu^2}{m_{\pi^-}^2}\right).$$

From the differential decay rate,

$$d\Gamma = \frac{1}{2m_{\pi^-}} |\mathcal{T}|^2 \frac{|\mathbf{p}|}{16\pi^2 m_{\pi^-}} d\Omega_{CM},$$

we get

$$\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu) = \frac{1}{8\pi} G_F^2 f_\pi^2 m_\mu^2 m_{\pi^-} \left(1 - \frac{m_\mu^2}{m_{\pi^-}^2}\right)^2. \tag{3}$$

Kaon decay

The interaction Lagrangian is similar for $K^- \to e^- \overline{\nu}_e$ and $K^- \to \mu^- \overline{\nu}_\mu$, so we use eq(3);

$$\Gamma(K^- \to e^- \overline{\nu}_e) = \frac{1}{8\pi} G_F^2 f_K^2 m_e^2 m_{K^-} \left(1 - \frac{m_e^2}{m_{K^-}^2}\right)^2$$

$$\Gamma(K^- \to \mu^- \overline{\nu}_\mu) = \frac{1}{8\pi} G_F^2 f_K^2 m_\mu^2 m_{K^-} \left(1 - \frac{m_\mu^2}{m_{K^-}^2}\right)^2$$

With $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, $m_e = 0.511$ MeV, $m_\mu = 0.106$ GeV, $m_{K^-} = 0.494$ GeV,

$$\Gamma(K^- \to e^- \overline{\nu}_e) \simeq 6.978 \times 10^{-28} f_K^2 \text{eV}^{-1}$$

$$\Gamma(K^- \to \mu^- \overline{\nu}_\mu) \simeq 2.732 \times 10^{-23} f_K^2 \text{eV}^{-1}$$

Thus, the ratio of the $K^- \to e^- \overline{\nu}_e$ and $K^- \to \mu^- \overline{\nu}_\mu$ decay rates is

$$\frac{\Gamma(K^- \to e^- \overline{\nu}_e)}{\Gamma(K^- \to \mu^- \overline{\nu}_\mu)} \simeq 2.554 \times 10^{-5},$$

which is close to 2.493×10^{-5} from the PDG.

Given that the lifetime of the K^- is $\tau = 1.2 \times 10^{-8}$ sec and the $K^- \to \mu^- \overline{\nu}_\mu$ branching ratio is 64%, we have

$$2.732 \times 10^{-23} f_K^2 \text{eV}^{-1} \cdot \frac{1}{1.055 \times 10^{-34} \text{J} \cdot \text{s}} \cdot \frac{1.6 \times 10^{-19} \text{J}}{\text{eV}} = \frac{1}{1.2 \times 10^{-8} \text{s}} \frac{64}{100}.$$

Therefore, $f_K \simeq 35.867$ MeV. Similar calculation for $\pi^- \to \mu^- \overline{\nu}_\mu$ gives $f_\pi \simeq 128.904$ MeV.

By the way, from the PDG $f_K \simeq 156.1$MeV, which is a lot different from the above value. This can be explained as follows. Actually, there is a $\sin\theta_C$ in the interaction Lagrangian for $K^- \to \mu^- \overline{\nu}_\mu$ and $\cos\theta_C$ for $\pi^- \to \mu^- \overline{\nu}_\mu$ where θ_C is the Cabbibo angle. These factors come from the coupling of $\overline{u}$ quark to $d' = d\cos\theta_C + s\sin\theta_C$ when considering the full theory. Generalization of this to all quakrs is encoded in the CKM matrix. Considering these, with $f_{K,\text{PDG}} = 156.1$MeV and $f_{\pi,\text{PDG}} = 130.41$ MeV, we get $f_{K,\text{here}} \simeq \sin\theta_C f_{K,\text{PDG}} = 35.358$ MeV and $f_{\pi,\text{here}} \simeq \cos\theta_C f_{\pi,\text{PDG}} = 127.016$ MeV. Above calculated results, $f_{K,\text{calculated here}} \simeq 35.867$ MeV and $f_{\pi,\text{calculated here}} \simeq 128.904$ MeV are close to these values.