Week 1 (due October 8)

1. Consider the group of motions of n-dimensional Euclidean space (or Euclidean group, for short). It is a Lie group which has as its Lie subgroups the rotation group $O(n)$ and the translation group $\mathbb{R}^n$. In fact, any element of the Euclidean group is a composition of rotation and translation. We can realize this group by matrices in the following way: to a transformation $x \mapsto Ux+v$, $U \in O(n)$, $v \in \mathbb{R}^n$ we assign a matrix $M(U,v)$ of size $(n+1)\times(n+1)$ given by

$$M(U,v) = \begin{pmatrix} U & v \\ 0 & 1 \end{pmatrix}.$$

(a) Show that this is indeed a matrix representation of the Euclidean group. Compute the commutation relations of the Lie algebra of the Euclidean group using this matrix representation.

(b) For any matrix Lie algebra we can define a scalar product as follows:

$$\langle v, w \rangle = \mathrm{Tr}(u \cdot v).$$

Here v, w are elements of the Lie algebra, and the dot denotes matrix multiplication. This is called the Killing metric, and it is invariant with respect to the action of the Lie group via the adjoint representation. (Do you see why?) Compute the Killing metric for the Lie algebra of the Euclidean group and show that it is degenerate.

2. Consider a system of two heavy quarks at rest. Compute the interaction energy due to a one-gluon exchange for the two possible irreducible $SU(3)$-representations of the system.

3. Let $\phi(x)$ be a complex scalar field in the defining representation of $SU(3)$ (i.e. ϕ is 3-component). Consider the following gauge-invariant Lagrangian for ϕ:

$$L_\phi = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi^\dagger \phi),$$

where $D_\mu \phi = \partial_\mu \phi - igA_\mu \phi$, and A_μ is the $SU(3)$ gauge field. The function V is an arbitrary real function of a real argument. Adding to this the Yang-Mills Lagrangian, we obtain a theory which may be called scalar QCD.

(a) Derive the equations of motion for scalar QCD. Show that the equation of motion for A_μ can be written in the form

$$D_\mu F^{\mu\nu} = -gJ^\nu$$

and compute J^μ in terms of ϕ and A_μ.

(b) Show that the current J^μ satisfies $D_\mu J^\mu = 0$, where D_μ is the covariant derivative. Do it in two different ways: first, show that it follows from the equation of motion for ϕ. Second, show that it follows tautologically from the equation $D_\mu F^{\mu\nu} = -gJ^\nu$, without assuming any particular form for J or any equation of motion for ϕ.

(c) Scalar QCD has a global symmetry

$$A \mapsto UAU^{-1}, \quad \phi \mapsto U\phi,$$

where U is a constant $SU(3)$ matrix. This symmetry arises from constant gauge transformations. Compute the Noether current corresponding to this symmetry. Does it coincide with J?